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# ON A CHARACTERIZATION USING RANDOM SUMS

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**Let $X_1$, $X_2$, and $X_3$ be independent random variables and let $Z_1 = X_1 + X_3$ and $Z_2 = X_2 + X_3$. It is known that if the characteristic functions of $X_k$, $k = 1, 2, 3$, do not vanish then the distribution of $(Z_1, Z_2)$ determines the distributions of $X_1$, $X_2$, and $X_3$ up to a shift. The aim of this paper is to prove a result of a similar nature using sums of a random number of random variables. We shall use $\sim$ for "has the same distribution as," r. v. for "random variable, " ch. f. for "characteristic function," and p. g. f. for "probability generating function."**

THEOREM 1. *Let $N, X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be independent* r.v.'s *where $X_n \sim X$, $Y_n \sim Y$, $n = 1, 2, \cdots$, and $X$ and $Y$ are nondegenerate real-valued* r.v.'s *having* ch.f.'s *$\varphi$ and $\psi$, respectively, which are of bounded variation on every finite interval. Let $N$ be a nonnegative integer-valued* r.v. *with* p.g.f.

$$Q(s) = p_0 + \sum_{n=1}^{\infty} p_n s^n , \qquad |s| \leqq 1 , \qquad p_n = P(N = n)$$

*and $0 < EN = m < \infty$. Assume that there is a neighborhood of 1 relative to the unit disk such that $Q^{-1}$ exists in this neighborhood. Denote*

$$U = 0 \text{ for } N = 0,\ U = X_1 + X_2 + \cdots + X_N \text{ for } N > 0 , \text{ and}$$
$$V = 0 \text{ for } N = 0,\ V = Y_1 + Y_2 + \cdots + Y_N \text{ for } N > 0 .$$

*Then the distribution of $(U, V)$ uniquely determines the distribution of $N$.*

*Proof.* Since $N, X_1, X_2, \cdots, Y_1, Y_2, \cdots$ are independent r.v.'s, the ch.f. of $(U, V)$, $\varphi_{(U,V)}$, satisfies the following:

$$\begin{aligned}
\varphi_{(U,V)}(r, t) &= E(e^{irU+itV}) \\
&= E(e^{irU+itV} \mid N = 0) \cdot P(N = 0) + \sum_{n=1}^{\infty} E(e^{irU+itV} \mid N = n) \cdot P(N = n) \\
&= E(1) \cdot p_0 + \sum_{n=1}^{\infty} E(e^{ir(X_1+\cdots+X_n)+it(Y_1+\cdots+Y_n)}) \cdot p_n \\
&= p_0 + \sum_{n=1}^{\infty} [E(e^{irX}) \cdot E(e^{itY})]^n \cdot p_n \\
&= p_0 + \sum_{n=1}^{\infty} [\varphi(r) \cdot \psi(t)]^n \cdot p_n \\
&= Q(\varphi(r) \cdot \psi(t)) , \qquad r, t \in R .
\end{aligned}$$