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THE BEST TWO-DIMENSIONAL DIOPHANTINE APPROXIMATION CONSTANT FOR CUBIC IRRATIONALS

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Let 1, β_1, β_2 be a basis of a real cubic number field K. Let $c_0 = c_0(\beta_1 \beta_2)$ be the infimum over all constants $c > 0$ such that

$$|q\beta_1 - p_1| < (c/q)^{1/2}, \qquad |q\beta_2 - p_2| < (c/q)^{1/2}$$

has an infinite number of solutions in integers $q > 0$, p_1, p_2. Set

$$C_0 = \sup_{\beta_1 \cdot \beta_2} c_0(\beta_1, \beta_2) .$$

The purpose of this note is to observe that combining a recent beautiful result in the geometry of numbers of A. C. Woods with the earlier work of the author, we obtain

THEOREM. **$C_0 = 2/7$.**

It is generally conjectured that the best 2-dimensional diophantine approximation constant is also 2/7 but the result here can only be taken as further evidence for the conjecture.

The statement that $C_0 \geqq 2/7$ is due to Cassels [2]. Moreover, it is shown in [1] that if 1, β_1, β_2 is the basis of a nontotally real cubic field K, then

$$c_0(\beta_1, \beta_2) \leqq 1/23^{1/2} < 2/7 .$$

Thus we may restrict our attention to totally real fields K. The following was also proved in [1]: for a full submodule $M \subseteqq K$ (a rank 3 free $\boldsymbol{Z}$-module) set

$$m_+(M) = \inf_{\substack{\xi \in M \\ \xi > 0 \\ N\xi > 0}} N\xi , \qquad m_-(M) = \inf_{\substack{\xi \in M \\ \xi > 0 \\ N\xi < 0}} |N\xi| ,$$

then

$$C_0^2 = \operatorname*{Sup}_{K,M} \frac{4m_+(M)m_-(M)}{D_M}$$

where D_M is the discriminant of M and $N = N_{\boldsymbol{Q}}^K$ is the norm from K to $\boldsymbol{Q}$. Thus it suffices to show that for all full modules M contained in a totally real cubic number field K, we have

$$m_+(M)m_-(M) \leqq \frac{D_M}{49} .$$