PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 1, 1980

MAPPING INTERVALS TO INTERVALS

M. B. ABRAHAMSE AND S. D. FISHER

We study the question of mapping intervals to intervals by rational functions which map the real line into the extended real line and the upper half plane into the upper half plane.

Let $\mathscr{R}$ be the set of rational functions which map the upper half plane into the upper half plane and the real line into the extended real line. A function in $\mathscr{R}$ is the upper half plane equivalent of a finite Blaschke product on the unit disk and is sometimes referred to as a rational Cayley inner function. Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers with $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ and let Λ be the set of points $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ in $\boldsymbol{R}^{2n}$ such that there is a φ in $\mathscr{R}$ with $\varphi([a_i, b_i]) = [A_i, B_i]$ for $i = 1, \cdots, n$. The purpose of this paper is to determine which points P lie in Λ. In other words, we wish to describe those collections of those intervals $[A_1, B_1], \cdots, [A_n, B_n]$ which are images under a rational Cayley inner function of the intervals $[a_1, b_1], \cdots, [a_n, b_n]$.

We note that it is always possible to map points to points, that is, there is a ψ in $\mathscr{R}$ with $\psi(a_i) = A_i$ and $\psi(b_i) = B_i$ for $i = 1, \cdots, n$. However, this function ψ may have a pole in some (a_j, b_j) and so $\psi([a_j, b_j]) \supseteqq \boldsymbol{R}$. The motivation for this research is the following question due to J. Rovnjak (verbal communication): Is it always possible to map intervals to intervals? In terms of the notation above, the question is whether Λ contains every point $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ with $A_i < B_i$ for $i = 1, \cdots, n$. The answer to this question will be shown to be in the negative. For example, we will show that if $n \geqq 2$, if $[A_i, B_i] \subset [a_i, b_i]$ for $i = 1, \cdots, n$, and if $[A_i, B_i] \neq [a_i, b_i]$ for some i, then there is no function φ in $\mathscr{R}$ with $\varphi([a_i, b_i]) = [A_i, B_i]$ for $i = 1, \cdots, n$.

The main result of this paper describes the set Λ, the closure of Λ, and the boundary of Λ in terms of functions in $\mathscr{R}$ with degree less than n and in terms of certain ideal points. This theorem is stated in §1 and three corollaries are established. The theorem is proved in §2 and some further observations are made in §3. We also include in §3 an elementary proof of the assertion mentioned above that it is always possible to map points to points; more general results can be found in [1; Article II].

We would like to point out that the analysis in this paper is similar in certain respects to that of the moment space of a Tchebycheff system as developed in [3]. In particular, the use of the