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GROUPS OF SQUARE-FREE ORDER ARE SCARCE

MICHAEL E. MAYS

We devise an upper bound for $B(n)$, the number of nonisomorphic groups whose orders are square-free and no larger than n, and a lower bound for $T(n)$, the number of nonisomorphic groups whose orders are no larger than n. It is then noted that $B(n) = o(T(n))$.

An open problem is to find a formula for $N(n)$, the number of nonisomorphic groups of order n. Balash [1] discovered such a formula in the special case where n is square-free, and Higman [4] and Sims [6] developed an asymptotic formula for the number of groups of order a power of a prime. In this paper we use Balash's result to determine an upper bound for $B(n)$, where

$$B(n) = \sum_{\substack{k \leqq n \\ k \text{ square-free}}} N(k) ,$$

and the work of Higman and Sims to bound $T(n)$, given by

$$T(n) = \sum_{k \leqq n} N(k) ,$$

from below.

Higman's result, as refined by Sims, is

LEMMA 1. *Let $A = A(n, p)$ be defined by $\log_p (N(p^n)) = An^3$. Then $A = 2/27 + O(n^{-1/3})$.*

Higman originally offered 2/27 as the function in the lower bound for A with error term $O(n^{-1})$ and 2/15 in the upper bound. Sims reduced the upper bound to $2/27 + O(n^{-1/3})$. The lower bound is all we need, and the constant is not important as long as it is positive.

THEOREM 1. *There exists a positive constant c such that*

$$T(n) \gg n^{c \log^2 n} .$$

Proof. Let $2^m < n \leqq 2^{m+1}$. Then for $n > 1$,

$$T(n) \geqq T(2^m) \gg 2^{km^3} \geqq n^{c \log^2 n} .$$

Murty and Murty [5] show that $T(n) \gg n \log\log\log n$, which is enough to conclude, with a result of Erdös and Szekeres [2], that abelian groups are scarce. They then ask about nilpotent groups. Their lower bound grows more slowly than n^2, whereas the bound