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UNITARY EQUIVALENCE TO INTEGRAL OPERATORS

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A bounded operator A on $L^2(X)$ is called an integral operator if there exists a measurable function k on $X \times X$ such that, for each f in $L^2(X)$,

$$\int |k(x, y)f(y)| d\mu(y) < \infty \quad \text{a.e.}$$

and

$$Af(x) = \int k(x, y)f(y)\, d\mu(y) \quad \text{a.e.}$$

(Throughout this paper, (X, μ) will denote a separable, σ-finite measure space which is not purely atomic.) An integral operator is called a Carleman operator if the inducing kernel k satisfies the stronger requirement:

$$\int |k(x, y)|^2 d\mu(y) < \infty \quad \text{for almost every } x \text{ in } X\,.$$

In a recent paper ([2]), V. B. Korotkov characterized those operators that are unitary equivalent to operators A on $L^2(X)$ such that both A and A^* are Carleman operators. It is natural to consider the same question with "Carleman" replaced by "integral." This question is settled by Theorem 2.

An integral operator is absolutely bounded if the representing kernel k is such that $|k|$ also induces a bounded integral operator. Theorem 3 furnishes three-fourths of a characterization of those operators that are unitarily equivalent to absolutely bounded integral operators. To be more specific, that theorem yields a necessary condition which is shown to be sufficient under an extra assumption. It is, however, the author's belief that this extra requirement is met by every bounded operator on a separable Hilbert space. This belief is formally stated as a conjecture at the end of the paper.

LEMMA 1. *Let A be a bounded operator on a separable Hilbert space $\mathscr{H}$. The following conditions on A are equivalent*:

(i) *There exists a Hilbert-Schmidt operator K such that $\ker(A-K) \cap \ker(A^* - K^*)$ is an infinite-dimensional subspace of $\mathscr{H}$*;

(ii) *There exists an orthonormal sequence $\{e_n\}_{n=1}^\infty$ in $\mathscr{H}$ such that $\|Ae_n\| \to 0$ and $\|A^*e_n\| \to 0$*;

(iii) *0 belongs to the essential spectrum of $A^*A + AA^*$.*

Proof. The implication (i) $\Rightarrow$ (ii) is proved by picking an orthonormal sequence $\{e_n\}_{n=1}^\infty$ in $\ker(A-K) \cap \ker(A^* - K^*)$ and recalling that K is compact.