ON INCOMPLETE POLYNOMIALS. II

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The approximation of \( x^n \) by incomplete polynomials is studied, i.e., we consider the extremal problem

\[
E_{n-k,k} = \inf \left\{ \left\| x^n + \sum_{j=1}^{k} d_j x^{n-j} \right\|_{[0,1]} : (d_1, \ldots, d_k) \in R^k \right\}, \quad n \geq k,
\]

for the supremum norm on \([0, 1]\). We show that, for \( k \) fixed, \( n^k E_{n-k,k} \to \varepsilon_k \) as \( n \to \infty \), where

\[
\varepsilon_k = \inf \left\{ \left\| e^{-t} \left( t^k + \sum_{j=0}^{k-1} a_j t^j \right) \right\|_{[0,\infty]} : (a_0, \ldots, a_{k-1}) \in R^k \right\}.
\]

A generalization of this result for the case of lacunary polynomial approximation is given, as well as inequalities for \( E_{n-k,k} \) and \( \varepsilon_k \). Furthermore, we prove that for any polynomial \( P(t) \) of degree at most \( k \), there holds for the supremum norm \( \| e^{-t} P(t) \|_{[0,\infty]} = \| e^{-t} P(t) \|_{[0,2\pi]} \).

1. Introduction. In this note, we continue our investigation [6], [7], [8], [3], of incomplete polynomials, a subject first introduced by G. G. Lorentz [4]. Following the notation of [7], if \( \pi_n \) denotes the set of all real polynomials of degree at most \( n \), then for each pair \((s, k)\) of nonnegative integers, \( \pi_{s+k} \) denotes the set of polynomials

\[
\pi_{s+k} = \{ x^s q(x) : q(x) \in \pi_k \},
\]

so that \( \pi_{s+k} \subset \pi_{s+k} \). A polynomial in \( \pi_{s+k} \) is called an incomplete polynomial of type \((s, k)\). For any set \( K \subset R \), the norm \( \| \cdot \|_K \) shall denote the supremum norm on \( K \), i.e., \( \| g \|_K = \sup \{|g(x)| : x \in K\} \). Setting

\[
E_{s+k} = \inf \{ \| x^s q(x) - q(x) \|_{[0,1]} : q(x) \in \pi_{k-s} \}, \quad \pi_{-1} = \{ 0 \},
\]

it is known [7] that, for each pair \((s, k)\), there exists a unique monic polynomial \( Q_{s,k}(x) \in \pi_{s+k} \) of exact degree \( s + k \), such that \( \| Q_{s,k} \|_{[0,1]} = E_{s+k} \).

In a recent paper, Borosh, Chui, and Smith [1] established that for any positive integer \( k \), there exist positive constants \( \sigma_1(k) \) and \( \sigma_2(k) \) such that

\[
\sigma_1(k) \leq n^k E_{n-k,k} \leq \sigma_2(k), \quad \forall n > k.
\]

They also proved that the coefficients of the extremal polynomials \( Q_{n-k,k}(x) \) are bounded as \( n \to \infty \).

One aim of this note is to derive (cf. (3.3)) explicit upper and