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ON INCOMPLETE POLYNOMIALS. II

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The approximation of x^n by incomplete polynomials is studied, i.e., we consider the extremal problem

$$E_{n-k,k} = \inf\left\{\left\|x^n + \sum_{j=1}^{k} d_j x^{n-j}\right\|_{[0,1]} : (d_1, \cdots, d_k) \in \boldsymbol{R}^k\right\}, \quad n \geqq k,$$

for the supremum norm on [0, 1]. We show that, for k fixed, $n^k E_{n-k,k} \to \varepsilon_k$ as $n \to \infty$, where

$$\varepsilon_k = \inf\left\{\left\| e^{-t}\left(t^k + \sum_{j=0}^{k-1} a_j t^j\right)\right\|_{[0,+\infty)} : (a_0, \cdots, a_{k-1}) \in \boldsymbol{R}^k\right\}.$$

A generalization of this result for the case of lacunary polynomial approximation is given, as well as inequalities for $E_{n-k,k}$ and ε_k. Furthermore, we prove that for any polynomial $P(t)$ of degree at most k, there holds for the supremum norm $\|e^{-t}P(t)\|_{[0,+\infty)} = \|e^{-t}P(t)\|_{[0,2k]}$.

1. Introduction. In this note, we continue our investigation [6], [7], [8], [3], of incomplete polynomials, a subject first introduced by G. G. Lorentz [4]. Following the notation of [7], if π_n denotes the set of all real polynomials of degree at most n, then for each pair (s, k) of nonnegative integers, $\pi_{s,k}$ denotes the set of polynomials

$$\pi_{s,k} := \{x^s q_k(x) : q_k \in \pi_k\}, \tag{1.1}$$

so that $\pi_{s,k} \subset \pi_{s+k}$. A polynomial in $\pi_{s,k}$ is called an *incomplete polynomial* of type (s, k). For any set $K \subset \boldsymbol{R}$, the norm $\|\cdot\|_K$ shall denote the supremum norm on K, i.e., $\|g\|_K := \sup\{|g(x)| : x \in K\}$. Setting

$$E_{s,k} := \inf\{\|x^s(x^k - q(x))\|_{[0,1]} : q \in \pi_{k-1}\}, \quad \pi_{-1} := \{0\}, \tag{1.2}$$

it is known [7] that, for each pair (s, k), there exists a unique monic polynomial $Q_{s,k}(x) \in \pi_{s,k}$ of exact degree $s + k$, such that $\|Q_{s,k}\|_{[0,1]} = E_{s,k}$.

In a recent paper, Borosh, Chui, and Smith [1] established that for any positive integer k, there exist positive constants $\sigma_1(k)$ and $\sigma_2(k)$ such that

$$\sigma_1(k) \leqq n^k E_{n-k,k} \leqq \sigma_2(k), \quad \forall n > k. \tag{1.3}$$

They also proved that the coefficients of the extremal polynomials $Q_{n-k,k}(x)$ are bounded as $n \to \infty$.

One aim of this note is to derive (cf. (3.3)) explicit upper and