PACIFIC JOURNAL OF MATHEMATICS
Vol. 92, No. 1, 1981

TRANSITIVE GROUPS OF ISOMETRIES ON H^n

ESTHER PORTNOY

This paper investigates transitive groups of direct isometries, without fixed points, of hyperbolic n-space H^n. For $n = 2$ there is a natural one-to-one correspondence between the set of all such groups and the set of ideal points of H^2. For $n \geqq 3$ there is an analogous collection of groups, which are in several senses the simplest but not the only such groups.

The existence of a transitive group of transformations without fixed points can be used to define an addition of points in the transformed space. The idea of sums of points in hyperbolic spaces has been used in probabilistic applications, for example by Kifer and by Karpelevich, Tutubalin and Shur. These involve a composition of measures based on the collection (not a group) of translations of H^2 or H^3. The group structure seems necessary for certain statistical questions, such as characterizations of normal distributions, which were in part the motivation for this investigation.

1. Some properties of H^n.

Theorems about hyperbolic geometry can be stated and proved in a variety of settings, and some ideas are more manageable in one setting than in another. This introductory section reviews the basic properties used in the rest of the paper, grouped according to the setting in which they seem most natural. Proofs are omitted, except for a few hints where the references may not provide the needed generality.

The earliest setting for hyperbolic geometry is the synthetic or axiomatic approach. The classical works of Lobachevsky and Bolyai are still quite readable; both are included in Bonola [1]. A good recent textbook, although limited to plane geometry, is Gans [6]. Eves [5], Chapter 7, is rather brief but does suggest some very useful relationships to topics covered elsewhere in the book. The extension to higher dimensions is actually not difficult; the very brief treatment of H^3 in many standard textbooks (for example, Kulczycki [9], pg. 110-124), together with the standard results for E^n, should be sufficient to point the way.

In this synthetic setting, two lines (or planes of any dimension) are called parallel if they are nonintersecting but asymptotic. A pencil of mutually parallel lines defines an ideal point of H^n, and we may say that the ideal point lies on each of the lines, or that each of the lines passes through or contains the ideal point. Two parallel planes have only one ideal point Ω in common, and they are said to be parallel at Ω. The term "hyperparallel" is used in this