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SOME EXACT SOLUTIONS OF THE NONLINEAR PROBLEM OF WATER WAVES

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Our purpose here is to present two related, strictly constructive methods for proving existence and uniqueness of certain steady problems of water waves. By "water waves", we mean free-surface flows, under gravity, of inviscid, irrotational, incompressible fluids.

Our results are related to those of Gerber [3], Moiseev [5], Krasovskii [4], Beckert [2] and others, and, in fact, include all the earlier works but Krasovskii's. Moreover, unlike any other work we know of, ours makes no use of complex function theory. Thus, our methods also apply, at least in principle, to three-dimensional flows. However, there appear to be serious technical difficulties with the generalization to three dimensions, and we reserve all discussion of this for later work.

Before discussing our results, we state the problem precisely. For this, choose a coordinate system with the Y-axis pointing up. Then, since the flow is assumed steady, we suppose the fluid occupies a domain $-B(X) < Y < T(X)$, independent of time, and we seek [10] a velocity potential Φ satisfying Laplace's equation

$$\Phi_{XX} + \Phi_{YY} = 0 \text{ for } -B(X) < Y < T(X)\,, \tag{1.1}$$

with two conditions on the (unknown) free surface $Y = T(X)$ and one on the (given) bottom $Y = -B(X)$. When $Y = T(X)$, the boundary conditions are

$$\Phi_Y = T_X \Phi_X\,, \tag{1.2}$$

and

$$2gT + \Phi_X^2 + \Phi_Y^2 = \text{constant}; \tag{1.3}$$

here, g denotes the acceleration due to gravity. Subscripts denote partial differentiation. On the bottom, $Y = -B(X)$, we require

$$\Phi_Y + B_X \Phi_X = 0\,. \tag{1.4}$$

In addition, two parameters are given, say, the mean depth and a mean speed.

Our main hypothesis is that the bottom is not too far from being flat and horizontal. We impose this condition by supposing that B has the form