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BOUNDS FOR THE PERRON ROOT OF A NONNEGATIVE IRREDUCIBLE PARTITIONED MATRIX

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It is well-known that the Perron root of a nonnegative irreducible matrix lies between the smallest and the largest row sum of A. This result is generalized to the case when the matrix A is partitioned into blocks.

1. Introduction and notations. If $A = (a_{ij})$ is a nonnegative irreducible $n \times n$ matrix, then the Perron root $r(A)$ of A satisfies the classical inequalities of Frobenius [1, p. 37; 9; 10, p. 63; 21, p. 31]

$$(1) \qquad \min_i S_i \leqq r(A) \leqq \max_i S_i ,$$

where S_i denotes the ith row sum of A, i.e., $S_i = \sum_{j=1}^n a_{ij}$ $(i = 1, \cdots, n)$. Moreover, we have strict inequalities in (1) unless all the S_i's are equal.

Other bounds for $r(A)$ have been found by Ledermann [13], Ostrowski [15], Brauer [2], Ostrowski and Schneider [17], Hall and Porsching [11], Brauer and Gentry [3; 4], and Deutsch [8]. (In some of these papers one has assumed that A is a positive matrix.)

The purpose of this paper is to give some simple generalizations of the inequalities (1), by considering certain partitionings of A.

We introduce a few notations. By $\boldsymbol{R}^m$ we denote the vector space of all column m-tuples of real numbers and $(x)_i$ denotes the ith (scalar) component of the vector $x \in \boldsymbol{R}^m$. By $\boldsymbol{R}^{m\times m}$ we denote the algebra of all $m \times m$ real matrices and $(A)_{ij}$ denotes the (scalar) (i, j)-entry of the matrix $A \in \boldsymbol{R}^{m\times m}$. For two vectors $x, y \in \boldsymbol{R}^m$, the inequality $x \leqq y$ $(x < y)$ means $(x_i) \leqq (y)_i$ $((x)_i < (y)_i)$ for all $i = 1, \cdots, m$. If $X_1, \cdots, X_t \in \boldsymbol{R}^{m\times m}$, then $\bigwedge_{s=1}^t X_s$ $(\bigvee_{s=1}^t X_s)$ denotes the greatest lower bound (least upper bound) of the matrices $X_1, \cdots, X_t$ in the natural (i.e., componentwise) partial ordering of $\boldsymbol{R}^{m\times m}$. In other words,

$$\left(\bigwedge_{s=1}^t X_s\right)_{ij} = \min_{s=1,\cdots,t} (X_s)_{ij} , \quad \left(\bigvee_{s=1}^t X_s\right)_{ij} = \max_{s=1,\cdots,t} (X_s)_{ij} ,$$

for all $i, j = 1, \cdots, m$.

The transpose of a matrix A (vector u) will be denoted by $A^{\top}$ $(u^{\top})$ and the Perron root of a nonnegative matrix $A \in \boldsymbol{R}^{m\times m}$ will be denoted by $r(A)$.

2. Let