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SUBBASES, CONVEX SETS, AND HYPERSPACES

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We introduce a notion of topological convexity. Main topics are: compactness of convexity structures, continuity of convex closure operators, and characterization of convex sets.

1. Closed subbases. It will be assumed that all spaces are T_1. We agree to use the word "subbase" for "closed subbase".

A subbase $\mathscr{S}$ of a space X is called a T_1 *subbase* if for each $S \in \mathscr{S}$ and for each $x \in X - S$ there is an $S' \in \mathscr{S}$ with $x \in S' \subset X - S$. $\mathscr{S}$ is called a *normal subbase* if for each pair of sets $S_1, S_2 \in \mathscr{S}$ with $S_1 \cap S_2 = \varnothing$ there exist $S_1', S_2' \in \mathscr{S}$ such that

$$S_1 \subset S_1' - S_2';\ S_2 \subset S_2' - S_1';\ S_1' \cup S_2' = X .$$

S_1' and S_2' are then said to *separate* (or: to *screen*) S_1 and S_2. Finally, $\mathscr{S}$ is called a *binary subbase* if each linked system $\mathscr{S}' \subset \mathscr{S}$ (i.e., a subcollection $\mathscr{S}'$ of $\mathscr{S}$ of which any two members meet) satisfies $\bigcap \mathscr{S}' \neq \varnothing$.

It is well-known that a binary subbase is T_1 (cf. van Mill [9, Lemma 1]), that a space carrying a binary subbase is compact (use Alexander's lemma), and that a space is completely regular iff it admits a normal T_1 (sub)base (cf. Frink [6, Thm. 1]; de Groot and Aarts [1, Thm. 2]).

2. Topological convexity structures. The analysis of convex sets in Euclidean space has led to the introduction of axiomatic convexity theory. One of the main purposes of this theory is to investigate in an abstract setting the relationship between various convexity invariants, modelled after famous theorems of Caratheodory, Helly and Radon on convex sets of $\boldsymbol{R}^n$.

In this paper we will study convexity structures compatible with a topological structure. Our purpose is not to study the above mentioned invariants, as we do not expect the introduction of a topology to give rise to new relationships in general. Instead, we are mainly concerned with the interaction between the two structures. Some of the main results are summed up at the end of this section. More results and applications can be found in [13], [14], [15], and [18].

In [7, p. 471], Kay and Womble define a *convexity* (*structure*) as a pair $(X, \mathscr{C})$, where $\mathscr{C}$ is a collection of subsets of X, such that $\varnothing, X \in \mathscr{C}$ and $\bigcap \mathscr{C}' \in \mathscr{C}$ for each nonempty family $\mathscr{C}' \subset \mathscr{C}$. $\mathscr{C}$ is also called a *convexity structure for* (*on*) X, and the members of $\mathscr{C}$