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CLOSED FACTORS OF NORMAL Z-SEMIMODULES

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Let M be a set of positive integers which is closed under multiplication and division whenever possible: if $m, n \in M$ and $m \mid n$, then $n/m \in M$. A *closed factor* of M is a subset $K \subset M$ which is closed under multiplication and for which there is another subset $R \subset M$ such that every member of M is uniquely representable as a product kr with $k \in K$ and $r \in R$. A theory is developed for determining all closed factors of a given M. The theory can be adapted to an analogous problem for convex polyhedral cones.

1. Introduction. The *factorization problem* for a set S with a binary operation $\circ$ can be stated as follows: Determine all pairs of subsets A, B of S such that each member of S is uniquely representable in the form $a \circ b$, $a \in A$, $b \in B$. More generally, if the operation is associative, one can replace the pair (A, B) with a sequence $(A_1, \cdots, A_n)$ of subsets of S.

Several authors have considered this problem for finite abelian groups. A special case, involving only subsets of a certain form, was solved by Hajós in the course of settling a classical conjecture of Minkowski on linear forms. (See [10] for a good exposition of this.) Subsequent work on factorizations of finite abelian group was done by Hajós, Rédei, Sands, and deBruijn. (References appear in [10].) Even for finite cyclic groups, the general factorization problem is unsolved. The corresponding problem for the infinite cyclic group was settled in a negative sense by Swenson in 1974 [12]. Partial results had previously been obtained by deBruijn [1], [3].

In [5], Long characterized all factorizations of the set $\{0, 1, \cdots, n-1\}$ under addition. The corresponding problem for certain subsets of the plane was studied by Stein [11] and Hansen [4].

Complete solutions to the factorization problem have been obtained for certain semigroups. In [2], deBruijn determined all factorizations of the additive semigroup of nonnegative integers. The two-dimensional version of this, in which S is the additive semigroup of nonnegative lattice points in the plane, was solved by Niven [9]. In [6], this author solved the n-dimensional version for all n, including infinite-dimensional cases: i.e., for any free commutative monoid. These results were extended in [7] to include certain submonoids of a free commutative monoid.

The results obtained in [2], [6], [7] and [9] can be summarized by saying that every factorization of one of these semigroups can