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FANS, REAL VALUATIONS, AND HEREDITARILY-PYTHAGOREN FIELDS

BILL JACOB

In this paper we give an explicit description of valuation rings compatible with certain infinite preprimes of a field. These results are essentially constructive versions of the results of L. Brocker and E. Becker relating fans and valuations. We discuss a number of examples in detail, including the higher orderings recently introduced by E. Becker. One of several applications is a generalization of the theorem of Brocker-Brown charactorizing superpythagorean fields.

1. The main theorem.

We begin by introducing the main definitions and notation of this subject. Let K be any field.

DEFINITION 1. (Harrison [7].) If $P \subseteqq K$ satisfies $-1 \notin P$, $P + P \subseteqq P$, $P \cdot P \subseteqq P$, then P called a *preprime* of K. In case $1 \in P$, P is called an *infinite* preprime of K. The maximal preprimes of K are called the Harrison primes of K.

Harrison primes were introduced as a possible generalization to arbitrary fields of the notion of a "prime" that arises in algebraic number fields. Throughout this paper we shall be concerned only with infinite preprimes. Following E. Becker [1], [2], [3] we give:

DEFINITION 2. An infinite preprime P is called a *preordering* if $P^{\cdot} = P - \{0\}$ is a subgroup of $K^{\cdot}$. A preordering P is called a *fan*, if whenever $U \subseteqq K^{\cdot}$ is a subgroup with $P^{\cdot} \subseteqq U$ and $-1 \notin U$, $U \cup \{0\}$ is a preorder of K. Finally, a preorder P is said to be *complete* if whenever $a^2 \in P$ it happens that $a \in P$ or $-a \in P$.

In [1], [2], [3] Becker shows that in many cases complete preorderings give rise to valuation rings. Very often, these complete preorderings are not Harrison primes. Thus it becomes interesting to know precisely when a preprime induces a valuation on a field. With this in mind we give:

DEFINITION 3. A preordering P is called a *strong fan* if whenever $a \notin \pm P$ it happens that $1 + a \in P \cup a \cdot P$. We shall call a strong fan P a *valuation fan* if in addition, whenever $a \notin \pm P$ but $1 + a \in P$, then $1 - a \in P$.