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JETS WITH REGULAR ZEROS

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If a mapgerm $f: \boldsymbol{R}^n, 0 \to \boldsymbol{R}^p, 0$ is a submersion ($rkf = p$), then its zero set is regular (the germ of a manifold) by the Implicit Function Theorem. Of course, there are also critical maps ($rkf < p$) whose zero sets are manifolds. Submersions have the added feature that one can discern that the zero set is regular from the first derivative of f at 0. Are there other instances in which one can tell purely from the derivatives of f at 0 that the zero set is regular? In this paper we show that there are, and go part way toward the eventual goal of describing them all.

The k-jet $j^kf(x)$ is $(f(x), Df(x), \cdots, D^kf(x))$ if $k < \infty$ or $(f(x), Df(x), \cdots)$ if $k = \infty$. A k-jet z is said to have regular zeros if every representative f (a germ such that $j^kf(x) = z$) has regular zero set. Suppose f has regular zero set V_f. In §2 we show that $j^\infty f$ has regular zeros iff f is ∞-$\mathscr{K}$-determined. In this case $\dim V_f = 0$ or $n - p$. If this dimension is 0, then f is ∞-$\mathscr{C}$-determined. If $p = 1$ and $\dim V_f = n - 1$, then $f = g \cdot h$ where g is a submersion and h is ∞-$\mathscr{C}$- determined. (If f is analytic, then it is ∞-$\mathscr{K}$-determined at x iff it is a submersion at each point of $V_f - \{x\}$ and is ∞-$\mathscr{C}$-determined at x iff $V_f = \{x\}$.) In §3 we show (again assuming V_f regular) that j^kf has regular zeros for some finite k iff f is ∞-$\mathscr{K}$-determined and either $\dim V_f = 0$ or $p = n - 1$. In this section we especially consider finitely $\mathscr{K}$-determined mapgerms. (If f is analytic, then it is finitely $\mathscr{K}$-determined at x iff it is a submersion at each of its complex zeros except possibly x.) Among the examples given are $x(x^2 + y^2)$, $x(x^2 + y^2)^2$ and $x(x^2 + y^2 + z^2)$; the first example is finitely determined and its 3-jet has regular zeros, the second is ∞- but not finitely determined and its 5-jet has regular zeros, and the third is ∞-determined and its ∞-jet but no finite jet has regular zeros.

For notational simplicity, we restrict our study of regular zeros to jets of germs at 0. Let $E_{n,p}$ denote the germs at 0 of C^∞ maps from $\boldsymbol{R}^n$ to $\boldsymbol{R}^p$, $m_{n,p}$ those which are 0 at 0, $E_n = E_{n,1}$ and $m_n = m_{n,1}$. Let $\mathscr{K}$ be the set of pairs (R, A), where $R \in m_{n,n}$ is invertible and A is a $p \times p$ matrix with entries in E_n such that $A(0)$ is invertible. Define a group structure on $\mathscr{K}$ by $(R', A') \cdot (R, A) = (R' \circ R, (A' \circ R)A)$ and a left action of $\mathscr{K}$ on $m_{n,p}$ by $(R, A) \cdot f = (Af) \circ R^{-1}$. Note that while this definition of $\mathscr{K}$ differs from that of Mather (see §2 of [6]), the $\mathscr{K}$ orbits are identical under both definitions. $\mathscr{R}$ and $\mathscr{C}$ are the subgroups in which A or R is the identity, respectively.