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NEW CONDITIONS FOR SUBNORMALITY

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The purpose of this paper is to establish some new characterizations of subnormality. One of these characterizations is interesting, in that the conditions are applied to "one vector at a time". This type of criterion is applied to show that verifying subnormality can be reduced to considering the restrictions of the operator to its cyclic invariant subspaces.

Denote the bounded linear operators on a separable Hilbert space H by $B(H)$. An operator $A \in B(H)$ is called subnormal if there exists an operator $N \in B(H \oplus H)$ so that N is a normal operator, $H \oplus 0$ is invariant for N and the restriction of N to $H \oplus 0$ equals A [8]. Some previous intrinsic characterizations of subnormality can be found in [2], [7], [8]. Also a summary of these results appears in [5].

An operator $T \in B(H)$ is called hyponormal if $T^*T - TT^* \geqq 0$. It is easy to see that T is hyponormal if and only if $\|Tx\| \geqq \|T^*x\|$ for all x in H. By a theorem of Douglas [6], this is equivalent to the existence of an operator $K \in B(H)$ satisfying $\|K\| \leqq 1$ and $T^* = KT$. This fact was explicitly brought to the author's attention in [3].

Now the subnormal operators comprise a subset of the hyponormal ones. Thus the question arises as to whether the contraction operator K relating T^* and T, as above, has properties which enable one to tell whether T is not only hyponormal, but subnormal as well. The following example shows that this is not the case. Let K, T, and S denote Toeplitz operators with symbols $\bar{z}^2$, $\bar{z} + z^3$, and z, respectively. (Here z stands for the identity function on the boundary of the unit disc.) Then $T^* = KT$ and $S^* = KS$, but S is subnormal and T is not [cf. 1]. The example for T comes from [4].

However if S is subnormal then so is S^n for $n = 0, 1, 2, \cdots$. Hence for $n = 0, 1, 2, \cdots$ there exist contractions $K_n \in B(H)$ with $S^{*n} = K_n S^n$. Also it is known that there are hyponormal operators T, which are not subnormal, with T^n hyponormal for $n = 0, 1, \cdots$ [13]. One might ask for conditions on the K_n guaranteeing that if $T^{*n} = K_n T^n$, $n = 0, 1, \cdots$, then T is subnormal. The following theorem provides these conditions.

THEOREM 1. *Let $T \in B(H)$. The following conditions on T are equivalent.*

(a) *T is subnormal.*