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ON THE SOLVABILITY OF BOUNDARY AND INITIAL-BOUNDARY VALUE PROBLEMS FOR THE NAVIER-STOKES SYSTEM IN DOMAINS WITH NONCOMPACT BOUNDARIES

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In the present paper the solvability of boundary value problems for the Stokes and Navier-Stokes equations is proved for noncompact domains with several "exits" to infinity. In these problems the velocity satisfies usual boundary conditions and has a bounded Dirichlet integral and the pressure has prescribed limiting values at infinity in some "exits".

1. **Preface.** It was shown by J. Heywood [1] that solutions of the Navier-Stokes system (even linearized) are not uniquely determined by the usual boundary and initial conditions in some domains with noncompact boundaries. It is connected with the possible noncoincidence of some spaces of divergence free vector fields defined in these domains. These spaces and linear sets of vector fields generating them are introduced as follows.

Let Ω be a domain in R^n, $n = 2, 3$, $\mathscr{C}_0^\infty(\Omega)$ — the set of all infinitely differentiable functions with compact supports contained in Ω, $\mathscr{J}_0^\infty(\Omega)$— the set of all divergence-free vector fields $\vec{u} \in \mathscr{C}_0^\infty(\Omega)$ (i.e., vector fields satisfying the equation $\nabla \cdot \vec{u} = \sum_{i=1}^n (\partial u_i/\partial x_i) = 0$), and $\overset{\circ}{W}{}_2^1(\Omega)$ and $\overset{\circ}{\mathscr{D}}(\Omega)$ — the completions of $\mathscr{C}_0^\infty(\Omega)$ in the norms $\|\vec{u}\|_{W_2^1(\Omega)} = \sqrt{(\vec{u}, \vec{u})^{(1)}}$ and $\|\vec{u}\|_{\mathscr{D}(\Omega)} = \sqrt{[\vec{u}, \vec{u}]}$ respectively, where $(\vec{u}, \vec{v})^{(1)} = \int_\Omega (\vec{u}\cdot\vec{v} + \vec{u}_x\cdot\vec{v}_x)\,dx$, $[\vec{u}, \vec{v}] = \int_\Omega \vec{u}_x\cdot\vec{v}_x dx$, $\vec{u}\cdot\vec{v} = \sum_{i=1}^n u_i v_i$, $\vec{u}_x\cdot\vec{v}_x = \sum_{i,j=1}^n (\partial u_i/\partial x_j)(\partial v_i/\partial x_j)$. Let $\mathscr{J}(\Omega)$ and $H(\Omega)$ be completions of $\mathscr{J}_0^\infty(\Omega)$ in these norms and $\hat{\mathscr{J}}(\Omega)$, $\hat{H}(\Omega)$ — the subspaces of all divegence-free vector fields in $\overset{\circ}{W}{}_2^1(\Omega)$ and $\overset{\circ}{\mathscr{D}}(\Omega)$. Clearly, $\hat{\mathscr{J}}(\Omega) \supset \mathscr{J}(\Omega)$ and $\hat{H}(\Omega) \supset H(\Omega)$. In [1] it is shown there are domains for which the quotient spaces $\hat{\mathscr{J}}(\Omega)/\mathscr{J}(\Omega)$, $\hat{H}(\Omega)/H(\Omega)$ are finite-dimensional, i.e., nontrivial (for instance, the domain $\Omega^0 = R^3\backslash S$, $S = \{x \in R^3\colon x_3 = 0, x_1^2 + x_2^2 \geqq 1\}$ possesses this property). A large class of such domains is found by O. Ladyzhenskaya, K. Piletskas and the author in [2, 3]. To describe the domains Ω considered in this paper, we define a standard domain $G \subset R^n$ given by the inequality

$$|z'| < g(z_n)\,, \quad z_n \geqq 0\,, \tag{1}$$

where $|z'| = |z_1|$ for $n = 2$, $|z'| = \sqrt{z_1^2 + z_2^2}$ for $n = 3$ and the function $g(t)$ satisfies the conditions