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C^*-ALGEBRAS ASSOCIATED WITH IRRATIONAL ROTATIONS

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For any irrational number α let A_α be the transformation group C^*-algebra for the action of the integers on the circle by powers of the rotation by angle $2\pi\alpha$. It is known that A_α is simple and has a unique normalized trace, τ. We show that for every β in $(\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$ there is a projection p in A_α with $\tau(p) = \beta$. When this fact is combined with the very recent result of Pimsner and Voiculescu that if p is any projection in A_α then $\tau(p)$ must be in the above set, one can immediately show that, except for some obvious redundancies, the A_α are not isomorphic for different α. Moreover, we show that A_α and A_β are strongly Morita equivalent exactly if α and β are in the same orbit under the action of GL (2, $\boldsymbol{Z}$) on irrational numbers.

0. Introduction. Let α be an irrational number, and let S denote the rotation by angle $2\pi\alpha$ on the circle, $\boldsymbol{T}$. Then the group of integers, $\boldsymbol{Z}$, acts as a transformation group on $\boldsymbol{T}$ by means of powers of S, and we can form the corresponding transformation group C^*-algebra, A_α, as defined in [8, 19, 30]. If we view S as also acting on functions on $\boldsymbol{T}$, and if $C(\boldsymbol{T})$ denotes the algebra of continuous complex-valued functions on $\boldsymbol{T}$, then S acts as an automorphism of $C(\boldsymbol{T})$. This gives an action of $\boldsymbol{Z}$ as a group of automorphisms of $C(\boldsymbol{T})$, and A_α is just the crossed product algebra for this action [19, 30]. A convenient concrete realization of A_α consists of the norm-closed *-algebra of operators on $L^2(\boldsymbol{T})$ generated by S together with all the pointwise multiplication operators, M_f, for $f \in C(\boldsymbol{T})$. It is known [8, 19, 22, 30] that A_α is a simple C^*-algebra (with identity element) not of type I, and that A_α has a unique normalized trace, τ. In fact, on the dense *-subalgebra $C_c(\boldsymbol{Z}, \boldsymbol{T}, \alpha)$ consisting of finite sums of the form $\Sigma M_{f_n} S^n$ the trace is given by

$$\tau(\Sigma M_{f_n} S^n) = \int_T f_0(t)dt ,$$

where dt is Lebesgue measure on the circle normalized to give the circle unit measure. (We remark that Theorem 1.1 of [27] can be used to show that this dense subalgebra itself is also simple.)

Little else has been known about the A_α. In particular, it has not been known whether or not the A_α are isomorphic as α varies. An interesting question raised in 7.3 of [8], and again recently in [22], is whether the A_α contain any projections. But in fact, shortly