PACIFIC JOURNAL OF MATHEMATICS
Vol. 94, No. 1, 1981

ON THE THEOREM OF S. KAKUTANI-M. NAGUMO AND J. L. WALSH FOR THE MEAN VALUE PROPERTY OF HARMONIC AND COMPLEX POLYNOMIALS

SHIGERU HARUKI

Let K be either the field of complex numbers C or the field of real numbers R. Let n be a fixed integer >2, and θ denote the number $\exp(2\pi i/n)$. Let $f, f_j\colon C \to K$ for $j = 0, \cdots, n$. Define Λ_n and Ω_n by

$$\Lambda_n(x, y) = n^{-1}\left[\sum_{j=0}^{n-1} f(x + \theta^j y)\right] - f(x) ,$$

$$\Omega_n(x, y) = n^{-1}\left[\sum_{j=0}^{n-1} f_j(x + \theta^j y)\right] - f_n(x) ,$$

for all $x, y \in C$. Our main result is the following. If $(n+1)$ unknown functions $f_j\colon C \to K$ for $j = 0, 1, \cdots, n$ satisfy the quasi mean value property $\Omega_n(x, y) = 0$ for all $x, y \in C$, then $(n+1)$ unknown functions f_j satisfy the difference functional equation $\Delta_u^n f_j(x) = 0$ for all $u, x \in C$ and for each $j = 0, 1, \cdots, n$, where the usual difference operator Δ_u is defined by $\Delta_u f(x) = f(x+u) - f(x)$. By using this result we prove somewhat stronger results than the theorem of S. Kakutani-M. Nagumo (Zenkoku, Sūgaku Danwakai, 66 (1935), 10–12) and J. L. Walsh (Bull. Amer. Math. Soc., 42 (1936), 923–930) for the mean value property $\Lambda_n(x, y) = 0$ of harmonic and complex polynomials.

1. Introduction. Throughout this note K denotes either the field of complex numbers C or the field of real numbers R. Let n be a fixed integer >2, and θ denote the number $\exp(2\pi i/n)$. Let $f, f_\nu\colon C \to K$ for $\nu = 0, 1, \cdots, n$. Define $\Lambda_n(x, y)$ and $\Omega_n(x, y)$ by

$$\Lambda_n(x, y) = n^{-1}\left[\sum_{\nu=0}^{n-1} f(x + \theta^\nu y)\right] - f(x) ,$$

$$\Omega_n(x, y) = n^{-1}\left[\sum_{\nu=0}^{n-1} f_\nu(x + \theta^\nu y)\right] - f_n(x)$$

for all $x, y \in C$. A function $f\colon C \to K$ is said to have the mean value property for polynomials if f satisfies the equation

$$\Lambda_n(x, y) = 0 \qquad \text{for all} \quad x, y \in C ,$$

while, as a generalization of the mean value property, $n+1$ functions $f_\nu\colon C \to K$ are said to have the quasi mean value property for polynomials if f_ν satisfy the equation

$$\Omega_n(x, y) = 0 \qquad \text{for all} \quad x, y \in C .$$