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INCOMPRESSIBILITY OF MAPS AND THE HOMOTOPY INVARIANCE OF ČECH COHOMOLOGY

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The Ω-compressibility dimension of a space Y is the largest integer r for which every map $f: X \to Y$ from a normal space with dimension less than r, the loop map $\Omega f: \Omega X \to \Omega Y$ is compressible. Bounds are determined for the Ω-compressibility dimension of Eilenberg-Maclane spaces of type $(Z, 2n)$ and $(Z_p k, 2n)$. In application this is used to settle the question as to when Čech cohomology based on finite covers is a homotopy invariant functor.

1. Compressibility: Background and statement results.

A map is called *compressible* if it is homotopic to a map whose image is contained in a compact subset of the target space. Let Ω denote the loop functor. A map $f: X \to Y$ is called *Ω-compressible* if $\Omega f: \Omega X \to \Omega Y$ is compressible.

Obviously if ΩY has the homotopy type of a compact space then any map into Y is Ω-compressible. On the other hand if ΩY has nonzero homology (or homotopy) groups in infinitely many dimensions then it is easy to see that the identity map $1_Y: Y \to Y$ is Ω-incompressible. In particular that will be the case when Y is a finite complex with nonzero (reduced) homology.

We shall be primarily concerned with the case $Y = K(G, n)$, an Eilenberg-Maclane space. Any essential map $f: S^{2n+1} \to K(Z, 2n+1)$ is Ω-incompressible since Ωf induces an isomorphism,

$$(\Omega f)^*: H^*(\Omega K(Z, 2n+1); Q) \longrightarrow H^*(\Omega S^{2n+1}; Q) ,$$

in rational cohomology. The following fundamental result is due to Weingram [11] (see also [9]).

WEINGRAM'S THEOREM. *For any finitely generated abelian group G, every essential map $f: S^{2n+1} \to K(G, 2n+1)$ is Ω-incompressible.*

For even dimensional spheres the situation is more complicated: if p is an odd prime then any map $S^{2n} \to K(Z_p k, 2n)$ is Ω-compressible, since S^{2n-1} is a H-space mod p (see [9] for details). Similarly, if $n = 1$, 2 or 4 then any map $S^{2n} \to K(G, 2n)$ is Ω-compressible for any G.

Any map from an n-dimensional (covering dimension) normal space into a CW-complex can be homotopically deformed into a map whose image is in the n-skeleton of the target space [see e.g., 6 p.