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CONDITIONS FOR BEING AN FGC DOMAIN

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A domain R is said to be FGC if every finitely generated R-module decomposes into a direct sum of cyclic submodules. The main result is: if R is a domain with quotient field Q, then R is FGC if and only if all of the following three conditions are satisfied: (1) R is Bezout, (2) Q/R is an injective R-module, and (3) there does not exist a continuous embedding of βN into $\operatorname{spec} R$ relative to the patch topology of $\operatorname{spec} R$. This result is also true if (3) is replaced: (3′) every nonzero element of R is an element of only finitely many maximal ideals of R. Using entire functions, there exists an example of a domain satisfying (1) and (2), but not satisfying (3). Also presented are some partial results towards generalizing the main result to commutative rings.

Introduction. All *rings* will be commutative with identity. R will always denote a ring. N will denote the set of all positive integers. Giving N the discrete topology, βN will denote the Stone-Cech compactification of N. Use $\operatorname{spec} R$ to denote the set of all prime ideals of R and $m \operatorname{spec} R$ to denote the set of all maximal ideals of R. For $a \in R$, use $V(a)$ for $\{P \in \operatorname{spec} R: a \in P\}$ and $D(a)$ for $\{P \in \operatorname{spec} R: a \notin P\} = \operatorname{spec} R - V(a)$. The *patch topology* of $\operatorname{spec} R$ is the topology which has $\{V(a)\}_{a \in R} \cup \{D(b)\}_{b \in R}$ as a subbasis of open sets. R is a *valuation ring* if the set of all the ideals of R forms a chain with respect to set inclusion (possibly R has zero-divisors). R is a *Bezout ring* if every finitely generated ideal of R is cyclic. We shall find it convenient to also use the following nonstandard notation: if $r \in R$, then $m \operatorname{spec}(r) = \{M \in m \operatorname{spec} R: r \in M\} = V(r) \cap m \operatorname{spec} R$. The main reference for this paper is [3], which includes characterizations of FGC rings and discussions of βN and the patch topology of $\operatorname{spec} R$ (although no homological algebra).

We next discuss the historical development of this subject to motivate the results. An R-module A is *linearly compact* if every family of cosets of submodules of A, that has the finite intersection property, has a nonempty intersection. A ring R is *maximal* if R is a linearly compact R-module. A ring R is *almost maximal* if R/I is a maximal ring for all nonzero proper ideals I of R. In 1952 I. Kaplansky [5] proved that if R is a valuation domain, then R is FGC if and only if R is almost maximal. In 1959 E. Matlis [6] proved that if R is a valuation domain with quotient field Q, then R is FGC if and only if Q/R is an injective R-module. One can restate these results as follows. If R is domain with only one