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A SUB-ELLIPTIC ESTIMATE FOR A CLASS OF INVARIANTLY DEFINED ELLIPTIC SYSTEMS

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We consider a certain invariantly defined nonlinear system of partial differential equations on a Riemannian manifold. Since a special case describes a steady, irrotational, compressible flow on the manifold, it is natural to refer to the (square of) the pointwise norm of the solution as the speed of the flow and to the density of the flow. Under appropriate restrictions on the density, the system is elliptic and we obtain a sub-elliptic estimate and a maximum principle for the speed of the flow in terms of the curvature of the manifold.

Introduction. Let M be an n-dimensional Riemannian manifold, and $\Lambda^p(M)$ the space of smooth p-forms on M. For $\omega \in \Lambda^p(M)$, $x \in M$, let $Q(\omega) = (\omega, \omega)(x) = {}^*(\omega \wedge {}^*\omega)(x)$ denote the pointwise norm of the form ω. Let $\rho\colon C^\infty(M) \to R$ be a given bounded smooth strictly positive function which we call the density function.

In the following, we consider the invariantly defined nonlinear system of equations for $\omega \in \Lambda^p(M)$:

$$\begin{aligned} d\omega &= 0 \\ \delta(\rho(Q(\omega))\omega) &= 0\,. \end{aligned} \tag{1}$$

If $p=1$, this system describes the motion of a compressible fluid on M and reduces to a single second order equation for the potential function. If the metric is Euclidean and $\rho(Q) = (1-(\gamma-1)/2Q)^{1/\gamma-1}$, it becomes the gas dynamics equation for polytropic flow in R^n. If $\rho \equiv 1$, one obtains the Laplace-Beltrami equation.

To be more explicit, if ω is a solution of (1), then it is also a solution of a homogeneous second order quasi-linear system, $A\omega = 0$. In local coordinates, let

$$\omega = \omega_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,, \quad \omega^{j_1\cdots j_p} = g^{j_1 i_1} \cdots g^{j_p i_p} \omega_{i_1\cdots i_p}\,,$$

and assume $\rho = \rho(Q(\omega))$. Then, $A\colon \Lambda^p(M) \to \Lambda^p(M)$ is given by

$$\begin{aligned} (A\omega)_{i_1\cdots i_p} = \sum_{i,j} g^{ji} \Big\{ &\rho \frac{\partial^2 \omega_{i_1\cdots i_p}}{\partial x^i \partial x^j} \\ &+ 2\rho' \omega^{j_1\cdots j_p} \sum_{k=1}^{p} \omega_{i_1\cdots i_{k-1} i i_{k+1}\cdots i_p} \frac{\partial^2 \omega_{j_1\cdots j_p}}{\partial x^{i_k} \partial s^j} \Big\} \\ &+ \text{lower order terms.} \end{aligned}$$

(We will observe the usual summation convention wherever possible.)