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THE VOLUME CUT OFF A SIMPLEX BY A HALF-SPACE

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A formula for the volume cut off an n-dimensional simplex by a half-space has immediate application in probability theory. This note presents a derivation of such a formula in a short and completely elementary way and also yields the moments of this volume about the coordinate hyperplanes, and the volumes cut off the lower dimensional faces of the simplex and their moments.

Before stating our results we introduce some notation. The "minus function" $(x)_-^k$ equals x^k if $x < 0$ and is zero otherwise. The nth divided difference of a real-valued function $f(x)$ is the symmetric function of $n+1$ arguments defined inductively by

$$D\{f(x)\colon x_0, x_1\} = \frac{f(x_0) - f(x_1)}{x_0 - x_1},$$

$$D\{f(x)\colon x_0, x_1, \cdots, x_n\} = \frac{D\{f(x)\colon x_0, x_1, \cdots, x_{n-1}\} - D\{f(x)\colon x_n, x_1, \cdots, x_{n-1}\}}{x_0 - x_n}.$$

If $x_0 = x_n$ we define

$$D\{f(x)\colon x_0, x_1, \cdots, x_{n-1}, x_0\} = \frac{\partial}{\partial x_0} D\{f(x)\colon x_0, x_1, \cdots, x_{n-1}\}$$

and similarly for all repeated arguments. The algorithm of Varsi [1, p 317] enables the calculation of such differences for $f(x) = (x)_-^k$ without the use of limiting processes. A proof of the symmetry and other elementary properties of divided differences may be found in [4].

The unit n-simplex with vertices $\boldsymbol{a}_0 = (0, \cdots 0)$, $\boldsymbol{a}_1 = (1, 0, \cdots, 0)$, $\cdots$, $\boldsymbol{a}_n = (0, \cdots, 0, 1)$ will be denoted by $\mathscr{A}_n$, and $\mathscr{B}_n$ will denote an n-simplex with vertices $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n$ where $\boldsymbol{b}_i$ has cartesian coordinates $(b_{i1}, \cdots, b_{in})$. We denote by $\mathscr{B}_k$ the k-face of $\mathscr{B}_n$ with vertices $\boldsymbol{b}_0$, $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_k$. The n-volume (n-dimensional volume) of $\mathscr{B}_n$ and the k-volume of $\mathscr{B}_k$, $k = 1, \cdots, n$ are given by

$$v_n(\mathscr{B}_n) = \frac{1}{n!}\begin{vmatrix} 1 & b_{01} \cdots b_{0n} \\ & \cdots \\ 1 & b_{n1} \cdots b_{nn} \end{vmatrix},$$