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ON THE LOCAL SPECTRUM AND THE ADJOINT

JAMES K. FINCH

Let X be a Banach space, X^* the dual space, and suppose that T is a closed linear operator on X. Assume that the domain of T is dense in X, so that the adjoint operator T^* is a closed linear operator on X^*. The local spectrum $\sigma(T, x)$ is defined below. In this paper we investigate some of the relations between $\sigma(T, x)$ and $\sigma(T^*, x^*)$. In particular we show that if $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are both empty, then $x^*x = 0$.

The resolvent function of T is $R_T(\lambda) = (\lambda I - T)^{-1}$; it is an operator valued function, and is defined and analytic for λ not in $\sigma(T)$, the spectrum of T. Setting $f_x(\lambda) = R_T(\lambda)x$, then f_x is analytic and satisfies $(\lambda I - T)f_x(\lambda) = x$ for all λ not in $\sigma(T)$. However, it may be possible to find analytic solutions to $(\lambda I - T)f(\lambda) = x$ for some (or all) values of λ that are in the spectrum of T. So we are led to define a *local resolvent function* of T at x as a vector valued analytic function f which satisfies $(\lambda I - T)f(\lambda) = x$. It is easily shown that for λ not in $\sigma(T)$, the only local resolvent is $f(\lambda) = R_T(\lambda)x$. But for λ in $\sigma(T)$, there may be more than one local resolvent function. The *local resolvent set* is the union of the domains of all the local resolvent functions. The point at infinity is included if there is a local resolvent function which is defined and bounded for $|\lambda| > r$. The *local spectrum* $\sigma(T, x)$ is the complement of the local resolvent set. Clearly $\sigma(T, x)$ is a closed subset of the spectrum $\sigma(T)$; it may be equal to $\sigma(T)$, properly contained in it, or even empty.

The operator T has the *single valued extension property* if there is at most one local resolvent function defined near any λ in $\boldsymbol{C}$; that is, whenever both f and g are local resolvent functions defined near λ, then $f = g$ there. In this case, there is a unique local resolvent with maximal domain. It can be shown that T has the single valued extension property iff $\sigma(T, x)$ is not empty for any $x \neq 0$. (Note that $\sigma(T, 0)$ is always empty; the local resolvent $f(\lambda) = 0$ is defined for all λ.)

If both T and T^* have the single valued extension property, then $\sigma(T, x) \cap \sigma(T^*, x^*) = \varnothing$ implies $x^*x = 0$ (see Lemma 1). This is not true if $\sigma(T, x) = \varnothing$ for some $x \neq 0$. For then $x^*x = 0$ for all x^*, from which it would follow that $x = 0$. Similarly, it is not true if $\sigma(T^*, x^*)$ is empty. But in Theorem 3 we show that if *both* $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are empty, then $x^*x = 0$.

Suppose that T does not have the single valued extension