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NEARLY STRATEGIC MEASURES

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Every finitely additive probability measure α defined on all subsets of a product space $X \times Y$ can be written as a unique convex combination $\alpha = p\mu + (1-p)\nu$ where μ is approximable in variation norm by strategic measures and ν is singular with respect to every strategic measure.

1. Introduction. For each nonempty set X, let $P(X)$ be the collection of finitely additive probability measures defined on all subsets of X. A *conditional probability* on a set Y given X is a mapping from X to $P(Y)$. *A strategy* σ on $X \times Y$ is a pair (σ_0, σ_1) where σ_0 is in $P(X)$ and σ_1 is a conditional probability on Y given X. Each strategy σ on $X \times Y$ determines a *strategic measure,* also denoted σ, in $P = P(X \times Y)$ by the formula

$$\sigma g = \iint g(x, y) d\sigma_1(y|x) d\sigma_0(x) ,$$

where g is a bounded, real-valued function on $X \times Y$. The collection Σ of all strategic measures was studied by Lester Dubins [3], who proved that, if X or Y is finite, then every member of P is *nearly strategic* in the sense that it can be approximated arbitrarily well in the sense of total variation by a strategic measure. However, Dubins also showed that if X and Y are infinite, then the collection $\bar{\Sigma}$ of all nearly strategic measures is a proper subset of P and, moreover, there exist elements in $\Sigma^{\perp}(=\bar{\Sigma}^{\perp})$, the set of measures in P singular with respect to every measure in Σ. (As usual, the finitely additive probability measures μ and ν are mutually singular if, for every positive ε, there is a set A such that $\mu(A) < \varepsilon$ and $\nu(A) > 1 - \varepsilon$.)

Here is our main result.

THEOREM 1. $\Sigma^{\perp\perp} = \bar{\Sigma}$.

This answers a question posed by Dubins in [3]. As Dubins pointed out, the following corollary is a consequence of Theorem 1 together with results of Bochner and Phillips [1].

COROLLARY 1. *Every μ in P can be written in the form*

$$\mu = p\sigma + (1 - p)\tau$$

with $\sigma \in \bar{\Sigma}$, $\tau \in \Sigma^{\perp}$, and $0 \leqq p \leqq 1$ where $p\sigma$, $(1-p)\tau$, and p are unique.