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CONCERNING THE MINIMUM OF PERMANENTS ON DOUBLY STOCHASTIC CIRCULANTS

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Let P_n be the permutation matrix such that $(P_n)_{ij} = 1$ if $j = i + 1 (\mathrm{mod}\, n)$. Minc [2] proved that the minimum of the permanent on the collection of $n \times n$ doubly stochastic circulants $\alpha I_n + \beta P_n + \gamma P_n^2$ is in $(1/2^n, 1/2^{n-1}]$, and if $n \geqq 5$ then the minimum is not achieved at $(1/3)I_n + (1/3)P_n + (1/3)P_n^2$. This paper proves that if $n \geqq 3$ then the minimum of such permanents is less than $1/2^{n-1}$, and if $n \in \{3, 4\}$ then this minimum is uniquely achieved at $(1/3)I_n + (1/3)P_n + (1/3)P_n^2$.

Introduction. Let n be a positive integer, let I_n denote the $n \times n$ identity matrix, and let P_n denote the full cycle permutation matrix such that $(P_n)_{ij} = 1$ if $j = i + 1 (\mathrm{mod}\, n)$. Minc [2] studied the permanent of circulants $\alpha I_n + \beta P_n + \gamma P_n^2$ and proved the following three theorems:

THEOREM 1. *If $n \geqq 3$ then*

$$\operatorname{per}(\alpha I_n + \beta P_n + \gamma P_n^2) = \left(\frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2}\right)^n + \left(\frac{\beta - \sqrt{\beta^2 + 4\alpha\gamma}}{2}\right)^n + \alpha^n + \gamma^n .$$

THEOREM 2. *If α, β, γ are nonnegative then*

$$\frac{1}{2^n} < \min_{\alpha+\beta+\gamma=1} \operatorname{per}(\alpha I_n + \beta P_n + \gamma P_n^2) \leqq \frac{1}{2^{n-1}} .$$

THEOREM 3. *If α, β, γ are nonnegative, $n \geqq 5$, then*

$$\min_{\alpha+\beta+\gamma=1} \operatorname{per}(\alpha I_n + \beta P_n + \gamma P_n^2) < \operatorname{per}\left(\frac{1}{3}I_n + \frac{1}{3}P_n + \frac{1}{3}P_n^2\right) .$$

MAIN RESULTS. Let $S = \{(\alpha, \gamma) \mid 0 \leqq \alpha, 0 \leqq \gamma, \alpha + \gamma \leqq 1\}$, and let f_n denote the function on S such that

$$f_n(\alpha, \gamma) = \operatorname{per}(\alpha I_n + (1 - \alpha - \gamma)P_n + \gamma P_n^2) .$$

THEOREM 4. *If $n \geqq 3$ then f_n is not minimum on the boundary of S.*

LEMMA TO THEOREM 4. *The minimum of f_n on the boundary of*