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A FIXED POINT THEOREM IN c_0

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It is proved that if K is the closed convex hull of a weakly convergent sequence in c_0 then each nonexpansive mapping $T: K \to K$ has a fixed point.

1. Introduction. The general problem with which we are concerned is: classify the weakly compact convex subsets K of a Banach space such that every nonexpansive mapping T of K into itself must necessarily have a fixed point. (T is said to be nonexpansive if $\|Tx - Ty\| \leqq \|x - y\|$ for all x and y in K.) We study this problem for the Banach space c_0.[1]

Section II is devoted to the proof of the theorem stated in the abstract, and § III to some extensions of it. For the present we wish to recall some known results in this area, and to explain why the space c_0 may be of special interest.

The problem posed above is of the following type: Let K be a subset of a locally convex topological vector space and $T: K \to K$ a mapping. Give conditions on K and T which insure T will have a fixed point.

The Tychonoff fixed point theorem [14] says if K is compact, convex and T is continuous then T has a fixed point. Banach's fixed point theorem [1] says if K is closed and a subset of a Banach space (more generally a complete metric space) and T is a strict contraction ($\|Tx - Ty\| \leqq \alpha\|x - y\|$ for all x, y in K and some $\alpha < 1$) then T has a unique fixed point.

Our problem may be viewed as combination of these two theorems. Note however that there is a strange feature in this combination: the condition on K concerns the weak topology while that on T concerns the norm topology. The seeming lack of connection between these conditions is what makes the problem so interesting and challenging.

From now on let us assume that K is a given convex weakly compact subset of a Banach space X and $T: K \to K$ is nonexpansive. Of course by translation one may assume $0 \in K$. Then for all $0 < r < 1$, $rT: K \to K$ and rT is a strict contraction. By the Banach theorem rT has a unique fixed point x_r and it is easily seen that $\|Tx_r - x_r\| \to 0$ as $r \to 1$. Thus there always exists a sequence of "approximate fixed points" for T. The points $\{x_r\}_{0 \leqq r < 1} (x_0 = 0)$ form a continuous curve in K. In fact it can be seen that if $0 < r <$

[1] D. Alspach [0] has recently given the first example of a weakly compact convex set K and a nonexpansive mapping on it without a fixed point.