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SOLVABILITY OF NONLINEAR OPERATOR EQUATIONS

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Let X and Y be Banach spaces, $P: X \to Y$ a Gateaux differentiable operator having closed graph. Suppose

(i) for each $R>0$ there is a $\delta>0$ such that

$$dP_x(B(0;1)) \supseteqq B(0;\delta) \text{ whenever } \|x\| \leqq R$$

(ii) $P^{-1}(K)$ is bounded whenever $\mathrm{cl}(K) \subseteqq Y$ is compact; then P is an open mapping of X onto Y. Similar results are obtained for compact Gateaux differentiable operators using a local version of (i); the same local version gives a domain invariance theorem for Gateaux differentiable operators having closed graph. Related results deal with M. Altman's theory of contractor directions and theory of normal solvability as developed by F. E. Browder and others.

1. Introduction.

Let X and Y be Banach spaces and $P: X \to Y$ a nonlinear operator. In this paper we consider the global implications of certain local assumptions on P and, in particular, derive general conditions under which P will be an open mapping of X onto Y. While our hypotheses are motivated by differentiability conditions on P, our results will apply to operators which need not even be continuous.

A 1959 theorem of R. S. Palais [17] provides the prototype for our results; Palais' theorem states that if $P: R^n \to R^n$ is a continuously differentiable mapping, then in order for P to be a diffeomorphism it is necessary and sufficient that

(1.1) $$0 \text{ is not an eigenvalue of } dP_x \text{ for any } x\,;$$

and

(1.2) $$\|Px\| \longrightarrow \infty \text{ as } \|x\| \longrightarrow \infty\,.$$

We are primarily interested in extending the surjectivity portion of Palais' conclusion to operators acting on arbitrary Banach spaces; our methods, in addition, will show the operators we consider are open maps.

Extensions of the above type have recently been obtained for continuously Fréchet differentiable operators P acting on a Banach space X (see § 3 for definitions). In [13] R. Kacurovskii shows that if (1.1) and (1.2) hold and if $(I-P)$ is completely continuous, then P is a homeomorphism of X onto X. M. Krasnoselskii has also observed this result is true [16], and in addition has observed that if (1.1) is strengthened to