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BOUNDARY POINTS OF JOINT NUMERICAL RANGES

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In this paper it is shown that the conical points of the joint numerical range belong to the joint spectrum. Moreover, we discuss the bare points and extreme points of the joint numerical ranges for the n-tuples of commuting normal operators and Toeplitz operators.

Introduction. The notion of the joint numerical range was first investigated by Halmos ([6], Prob. 166). Dash [4] tried to find how much of the knowledge about the numerical range in the single operator case carried over to the analogous situation in the case of an n-tuple of operators. Our purpose is to discuss the same subject as his. Dash [4] studied particularly about the convexity of the numerical range known as the Toeplitz-Hausdorff theorem. Here we shall, however, bring the boundary point of the numerical range into focus. In the case of a single operator, many authors have asserted the results referring to the relation between the numerical range and spectrum. Concerning these, Dash [4], Juneja [8], Abramov [1], Buoni and Wadhwa [3] have investigated the relation between the joint spectrum and joint numerical range. Abramov [1] has shown that the conical point of the closure of the joint numerical range of $A = (A_1, \cdots, A_n)$ belongs to the joint approximate point spectrum of A in the case of the family A consisting of self-adjoint operators. In §1, our result shall be given more clearly than Abramov's one even to the family of arbitrary operators, by means of Hildebrandt's technique [7]. In §2, we shall introduce a class of operator-families called *joint normaloid*. And, in §3, we shall discuss the bare points and extreme points of joint numerical ranges for the operator-families belonging to the joint normaloid.

Notation and definition. Throughout this paper, H will be a complex Hilbert space with the scalar product $(\, , \,)$ and the norm $\| \cdot \|$, and all operators on H will be assumed to be linear and bounded. Let $A = (A_1, \cdots, A_n)$ be an n-tuple of operators on H. The *joint numerical range* of A is the subset $W(A)$ of the n-dimensional unitary space C^n such that

$$W(A) = \{((A_1x, x), \cdots, (A_nx, x)) : x \in H, \|x\| = 1\} .$$

In the case of $n = 1$, it is the usual numerical range of an operator.