PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 2, 1981

SOME REMARKS ABOUT C^∞ VECTORS IN REPRESENTATIONS OF CONNECTED LOCALLY COMPACT GROUPS

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Given a continuous representation U of a connected locally compact group G in a quasi-complete locally convex topological vector space E, one may introduce the space E_∞ of C^∞-vectors which contains the dense space F_∞ of regular vectors. Natural questions are then: (1) does $F_\infty = E_\infty$ hold? (2) is the differential U_∞ of U a representation of the Lie algebra of G on E_∞? We here prove that answer to (1) is "yes" when G is a quotient of a direct product of compact connected Lie groups and E has a continuous norm, and that answer to (2) is always "yes". Of special interest are locally compact groups which are almost Lie in the sense that any subgroup algebraically generated by two continuous one-parameter subgroups is a Lie group in a finer connected topology. We prove that a connected locally compact group is almost Lie if and only if its universal covering in the sense of Lashof is $H\times A$ with H simply connected Lie group and A direct product of copies of R.

Let G be a connected locally compact group and $\{H_\alpha, \alpha\in I\}$ a directed decreasing family of normal compact subgroups of G such that

(1) $G_\alpha = G/H_\alpha$ is a Lie group for each $\alpha \in I$ (by a Lie group we shall always mean a finite dimensional real Lie group),

and

(2) $$\bigcap_{\alpha\in I} H_\alpha = \{e\}\,, \quad e \text{ identity of } G\,.$$

We shall identify G to the projective limit of the G_α's. Denote by $\mathfrak{G}$ the Lie algebra of G, which is the projective limit of the Lie algebras $\mathfrak{G}_\alpha$ of the Lie groups G_α. If $X = (X_\alpha) \in \mathfrak{G}$, $t \in \boldsymbol{R}$, denote by $\exp tX$ the element $(\exp tX_\alpha)$ of G. Let U be a continuous representation of G in a quasi-complete locally convex topological vector space E. For $\alpha\in I$, introduce $A_\alpha = \int_{H_\alpha} U(h)d\mu_\alpha(h)$, with μ_α normalized Haar measure of H_α. A_α is a continuous endomorphism of E ([2], Prop. 10(a), p. 17).

LEMMA 1.
(i) *For each $\alpha\in I$, A_α is a projector of E* (i.e., $A_\alpha^2 = A_\alpha$), *orthogonal if E is a Hilbert space and U unitary; its range is the*