PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 2, 1981

A GENERALIZATION OF A CLASSICAL NECESSARY CONDITION FOR CONVERGENCE OF CONTINUED FRACTIONS[1]

ROBERT HELLER AND F. A. ROACH

One of the most frequently cited necessary conditions for convergence of continued fractions is the divergence of a particular series. In this paper, we show that convergence of a continued fraction implies divergence of each member of an infinite collection of series.

We will be concerned with continued fractions which are of, or can be put into (cf. Wall [4], pp. 19-26), the form

$$b_0 + \frac{1}{b_1} {+} \frac{1}{b_2} {+} \frac{1}{b_3} {+} \cdots . \tag{1}$$

If we let

$$\begin{aligned} &A_0 = b_0\,, \quad A_1 = b_1 A_0 + 1\,, \quad B_0 = 1\,, \quad B_1 = b_1\,, \\ &A_p = b_p A_{p-1} + A_{p-2}\,, \quad \text{and} \\ &B_p = b_p B_{p-1} + B_{p-2}\,, \quad p = 2, 3, 4, \cdots , \end{aligned} \tag{2}$$

then the nth approximant of (1) is given by A_n/B_n. As is customary, we say that (1) converges provided that not infinitely many of the denominators B_p are zero and $\{A_p/B_p\}$ converges to a finite limit.

The principal result given in this paper is the following theorem.

THEOREM. *Suppose that u is a complex number, v is a complex number such that $-4 < uv \leqq 0$, and $u = 0$ if $v = 0$. If both $\sum |b_{2p-1} - u|$ and $\sum |b_{2p} - v|$ converge, then* (1) *diverges.*

Considering the case where $u = v$, we immediately have the following result.

COROLLARY. *In order for* (1) *to converge, it is necessary that for each real number k between -2 and 2, $\sum |b_p - ki|$ diverge.*

If $k = 0$, this becomes what is often referred to as von Koch's theorem. According to Perron [2] p. 235, it was first proved by Stern [3] in 1860; additional information concerning the numerators and denominators of the approximants was obtained by von Koch [1] in 1895 (cf. Wall [4], pp. 27-29).

[1] This paper is dedicated to the memory of Keith Heller.