PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 2, 1981

# DENSITIES AND SUMMABILITY

A. R. FREEDMAN AND J. J. SEMBER

**The ordinary asymptotic density of a set $A$ of positive integers is $\nu(A)=\lim_{n\to\infty}A(n)/n$, where $A(n)$ is the cardinality of the set $A\cap\{1, 2, \cdots, n\}$. It is known that the space of bounded strongly Cesàro summable sequences are just those bounded sequences that converge (in the ordinary sense) after the removal of a suitable collection of terms, the indices of which form a set $A$ for which $\nu(A)=0$. In this paper we introduce a general concept of density and then examine the relationship, suggested by the above observation, between these densities and the strong convengence fields of various summability methods. These include all nonnegative regular matrix methods as well as the famous nonmatrix method called almost convergence.**

The characterization of the bounded strongly Cesàro summable sequences mentioned above is significant in ergodic theory, where it relates to the study of weakly mixing transformations ([4], p. 38; [7], pp. 40–41).

The concept of a lower asymptotic density is presented axiomatically in § 2. Certain essential properties of these densities are proved and the "natural density" associated with the lower density is defined. The natural density has some of the properties of a measure but, in particular, is not a countably additive function. Of interest, therefore, are certain additivity properties (we call them (AP) and (APO)), valid for some natural densities, that are approximations to countable additivity.

Section 3 contains examples of densities. Of particular interest are those generated by nonnegative regular matrices, and another called uniform density.

In § 4 we investigate sequence spaces associated with a density. One such space is the space $\omega_\delta$ of "nearly convergent" sequences (Definition 4.2) and another is the strong summability field $|c_s|$ of a summability method $S$ that is "related" to the density in the sense of Definition 4.9. Whether or not the (APO) property holds for the density turns out to be crucial in the comparison of the sequence spaces $\omega_\delta$ and $|c_s|$.

We use the following notation: The set of positive integers will be denoted by $I$. For $A$, $B\subseteq I$, we write $A\sim B$ ($A$ is asymptotically equal to $B$) if the symmetric difference $A\Delta B$ is finite. For two sets $A$ and $B$, the set-theoretic difference is denoted by $A\backslash B = \{x: x\in A,\ x\notin B\}$. Let $\emptyset$ denote the empty set. Sequences of real