PACIFIC JOURNAL OF MATHEMATICS
Vol. 96, No. 1, 1981

INVARIANT MANIFOLDS FOR REGULAR POINTS

JORGE LEWOWICZ

In this article we prove, for a differentiable vector field or a diffeomorphism on a smooth manifold, that the set of points such that the semitrajectories issuing from them approach a particular semitrajectory at a given exponential rate, constitute a differentiable submanifold, provided the differential of the flow has a certain similar behavior on that trajectory. (See Theorem 1 below, for a precise statement). In particular, the stable manifold theorem for hyperbolic sets ([3], [6, XI]) follows as a corollary.

Although we only consider the C^1-case, the same methods, which are essentially classical ([2, Ch. XIII]), could be applied to obtain higher differentiability properties.

Since I have not seen in the literature this type of results for points which are neither equillibrium nor periodic points, and on account of [6, XI-8], I thought that their publication might not be entirely devoid of interest.

1. Terminology and notation are standard. If X is a differentiable vector field on a smooth manifold M, ϕ will always denote the corresponding flow, and ϕ_t the diffeomorphism $x \to \phi(x, t)$, $x \in M$, $t \in R$. For brevity, we shall sometimes write $x(t)$ or $y(t)$ instead of $\phi(x, t)$ or $\phi(y, t)$ respectively.

THEOREM 1. *Let M be compact smooth (C^∞) Riemannian manifold and X a C^1-vector field. Assume that for some $x \in M$, there are subspaces E, I; $E \oplus I = T_xM$, such that for some positive mumbers K, λ, μ, $\mu < \lambda$, we have*

$$(1) \qquad \|\phi'_s(x(t))e_t\| < Ke^{-\lambda s}\|e_t\| \quad \textit{for} \quad e_t \in \phi'_t(x)E,\ s,\ t > 0\ ,$$

and

$$(2) \qquad \|\phi_{-s}(x(t))i_t\| < Ke^{\mu s}\|i_t\| \quad \textit{for} \quad i_t \in \phi'_t(x)I,\ 0 < s < t\ .$$

Then, $W_\lambda(x) = \{y \in M / \overline{\lim}\,(1/t) \log \operatorname{dist}(\phi(y, t), \phi(x, t)) < -\lambda\}$ is a C^1-submanifold of M, such that $T_xW_\lambda(x) = E$.

Condition (1) means that ϕ'_t strongly contracts the bundle $\bigcup_{t>0} \phi'_t(x)E$, while (2), which is equivalent to

$$(2') \qquad \|\phi'_s(x(t))i_t\| \geqq He^{-\mu s}\|i_t\|\ , \quad t,\ s > 0$$

for some $H > 0$, only says that ϕ'_t does not contract as strongly on $\bigcup_{t>0} \phi'_t(x)I$.