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SUFFICIENCY, KMS CONDITION AND RELATIVE ENTROPY IN VON NEUMANN ALGEBRAS

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The sufficiency in von Neumann algebras is discussed with some applications to classification of normal states. It is shown that the concept of sufficiency characterizes the KMS-states and the invariant states with respect to a modular automorphism group. The relations between the sufficiency and the relative entropy are established.

Introduction. Since the investigation of sufficient statistics in abstract measure theoretic terms was initiated by Halmos and Savage [10], the concept of sufficiency has been developed by many mathematical statisticians in terms of various relations given by comparison of experiments, risk functions within the framework of statistical decision problems and so on. A characterization of sufficiency was given in [12] through the measure of Kullback-Leibler information.

The concept of sufficiency was first generalized by Umegaki [22, 23] to the noncommutative case of semi-finite von Neumann algebras with some extension of the Kullback-Leibler information (usually called the relative entropy). Later the related discussions especially concerning the relative entropy for quantum systems have been made by several authors, e.g., Araki [2, 3], Gudder and Marchand [7], and Lindblad [13].

As defined precisely and explained in §§ 1 and 4 of this paper, the concept of sufficiency is more or less considered through the informativity of a certain subalgebra with respect to a given algebra for a dynamical system of interest. Namely, in the case that such a subalgebra is sufficient, the relative entropy on the subalgebra is equal to that on the given algebra. This fact may or may not be a reason why the concept of sufficiency has not been entered into analysis of physical systems, in which the change of entropy is thought of more relevant.

The Kubo-Martin-Schwinger (KMS) condition was introduced by these three authors [11, 14] as a boundary condition of the thermal Green function. Haag, Hugenholtz and Winnink [8] showed that in the operator algebraic framework this condition is a fundamental one describing thermal equilibrium of quantum systems. The KMS condition through the Tomita-Takesaki theory now becomes a core of studying von Neumann algebras.

Under the above historical basis, our main motivation of this