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CLOSED ORBITS OF CONVEX SETS OF OPERATORS ON THE DISK ALGEBRA

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Let $\mathscr{P}$ denote the set of linear transformations of the disk algebra which have norm one and fix the constants. In this paper the orbits of functions in the disk algebra under the action of $\mathscr{P}$ are described.

Let $\mathscr{A}$ denote the disk algebra, i.e., the algebra of functions which are analytic on the open unit disk D and continuous on $\bar{D}$. Let $\mathscr{A}$ be equipped with the sup-norm $\|\cdots\|$ and let $\mathscr{U}$ denote the closed unit ball in $\mathscr{A}$. Consider the set $\mathscr{P}$ of linear operators which map $\mathscr{A}$ into itself, have norm 1 and fix the constants. $\mathscr{P}$ acts as a semi-group of transformations of the set $\mathscr{U}$. In this paper we study the closed orbits of functions in $\mathscr{U}$ under the action of $\mathscr{P}$, i.e., the sets

$$\mathscr{P}f = \text{closure } \{Tf \mid T \in \mathscr{P}\}$$

for $f \in \mathscr{U}$. We will show that $\mathscr{P}f$ is the closed convex hull of the functions $[F, G]f$, where F and G range over the inner functions in $\mathscr{A}$. Here

$$[F, G]g(z) = (2\pi n i)^{-1}\int_{\partial D} g(\xi)F'(\xi)(G(z) - F(\xi))^{-1}d\xi ,$$

where n is the number of zeros of F and $z \in D$. (Recall that the inner functions in $\mathscr{A}$ are exactly the finite Blaschke products, that is functions of the form

$$F(z) = e^{is} \prod_{j-1}^{n} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}$$

where $|\alpha_j| < 1$ for $j = 1, 2, \cdots, n$.) We will also show that our result can be viewed as a generalization of a theorem due to Fisher [3]. The final section of the paper contains a discussion of the possibility of extending our results to the algebra H^∞ of bounded analytic functions on D.

1. Background. Perhaps a few sentences should be devoted to the context of this work in the literature. In [9] Phelps asked for a description of the extreme points of the convex set $\mathscr{P}$. In the same paper he was able to characterize those extreme elements of $\mathscr{P}$ which happen to be multiplicative. They are exactly the