PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 2, 1981

POINTWISE DOMINATION OF MATRICES AND COMPARISON OF $\mathscr{I}_p$ NORMS

BARRY SIMON

Let p be a real number in $[1, \infty)$ which is not an even integer. Let $N = 2[p/2] + 5$. We give examples of $N \times N$ matrices A and B, so that $|a_{ij}| \leqq b_{ij}$ but $\mathrm{Tr}([A^*A]^{p/2}) > \mathrm{Tr}([B^*B]^{p/2})$.

Let A and B be $N \times N$ matrices with

$$|a_{ij}| \leqq b_{ij} . \tag{1}$$

If we define the p norm of a matrix by

$$\|A\|_p = \mathrm{Tr}([A^*A]^{p/2})^{1/p} \tag{2}$$

then it is trivial that, if p is an even integer, then

$$\|A\|_p \leqq \|B\|_p \tag{3}$$

when (1) holds. For one need only write out the trace explicitly in terms of matrix elements. In a more general context, we conjectured in [5] that (1) implies (3) whenever $p \geqq 2$. The attractiveness of this conjecture is shown by the fact that I know of at least five people other than myself who have worked on proving it.

It was thus quite surprising that Peller [3] announced that (3) fails for some infinite matrices whenever p is not an even integer. In correspondence, Peller described his counterexample which relies on his beautiful but elaborate theory of $\mathscr{I}_p$ Hankel operators (4) and on a paper of Boas (2). It follows from Peller's example that (3) must fail for some finite N but it is not clear for which N. Our purpose here is to give explicit N and to avoid the complications of Peller's $\mathscr{I}_p$-Hankel theory.

The idea of the construction is very simple. Boas [2] constructed polynomials $f(z)$, $g(z)$ with $\int |f(e^{i\theta})|^p d\theta > \int |g(e^{i\theta})|^p d\theta$ even though the coefficients, a_n, of f and coefficients, b_n, of g obey $|a_n| \leqq b_n$. a and b should be thought of as Fourier coefficients of $f(e^{i\theta})$ and $g(e^{i\theta})$. It is obvious that for sufficiently large N, $\sum_{j=0}^{N-1} |f(e^{ij\theta_N})|^p \geqq \sum_{j=0}^{N-1} |g(e^{ij\theta_N})|$ where $\theta_N = 2\pi/N$. Again f and g should be viewed as functions on Z_N and the coefficients of the polynomial (if N is larger than the degrees) as Z_N-Fourier components. But the functions on Z_N are naturally imbedded in $N \times N$ matrices in such a way $\|A\|_p^p$ is just $\sum |f(e^{ij\theta_N})|^p$ and so that the order (1) is equivalent to the order on Fourier coefficients.