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ORDERS OF FINITE ALGEBRAIC GROUPS

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Let $\bar{G}$ be a simply connected simple algebraic group over a finite field F_q of q elements. The order of the group $\bar{G}(F_q)$ of F_q-rational points of $\bar{G}$ is well-known (cf: Steinberg, Carter). The proof makes use of the Bruhat decomposition and the study of polynomials invariant under the action the Weyl group. In this paper we deduce the order of $\bar{G}(F_q)$ from an explicit formula for the integral $M(s, \Lambda)$ which occurs in Langlands' theory of Eisenstein series.

First of all, according to a theorem of Lang $\bar{G}$ is quasi-split (cf: Lang [9], Satake [13] p. 105) and from Steinberg's theorem (cf: Steinberg [14], Kneser [6] p. 255) $\bar{G}$ is either a Chevalley group or a twisted group of one of the following types: ${}^2A_l(l \geqq 2)$, ${}^2D_l(l \geqq 4)$, ${}^2E_6$, ${}^3D_4$, ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$. To simplify matters we shall assume that the characteristic of F_q is not 2 and 3 and exclude groups of the type ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$. Furthermore we can assume that there exists a quasi-split simple algebraic group G defined over a p-adic number field F such that the residue field of F is isomorphic to F_q, G splits over an unramified Galois extension E of F and G reduces modulo p to $\bar{G}$ (cf: Weil [17]).

1. Fix a Haar measure dx on F such that the volume of the ring R of p-adic integers in F is one. Let ω be a left invariant highest F-differential form on G. Then ω and dx determines a Haar measure on $G(F)$ which will also be denoted by ω (cf: Weil [17]).

LEMMA 1. *Let m be the dimension of $\bar{G}$ and $|\bar{G}(F_q)|$ be the order of $\bar{G}(F_q)$. Then*

$$|\bar{G}(F_q)| = q^m \int_{G(R)} \omega \,. \tag{1}$$

This is proved in Weil [17] p. 22.

2. Let B be a Borel subgroup of G defined over F and A a maximal torus of G in B. Then by assumption the Galois group $\mathrm{Gal}(E/F)$ acts on the group $X(A)$ of rational characters of A. This gives rise to a representation

$$\pi: \mathrm{Gal}(E/F) \longrightarrow \mathrm{Eng}\,(X(A) \otimes_{Z} Q)\,.$$