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REMARKS ON PIECEWISE-LINEAR ALGEBRA

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This note studies some of the basic properties of the category whose objects are finite unions of (open and closed) polyhedra and whose morphisms are (not necessarily continuous) piecewise-linear maps.

Introduction. A function $f: V \to W$ between real vector spaces is *piecewise-linear* (PL) if there exists a partition of V into "open polyhedra" X_i (i.e., relative interiors of polyhedra) such that f is affine on each X_i. (As distinct to the case of PL-topology, no continuity is required of f.) Images and preimages under PL-maps give rise to finite unions of open polyhedra, or PL-*sets*; conversely PL functions can be characterized by the fact that their graphs are PL-sets. This paper studies some basic algebraic properties of the category $\boldsymbol{PL}$, proving in particular that it is an exact category, and in fact a pretopos. A classification is given for the isomorphism classes of objects of $\boldsymbol{PL}$, in terms of a two-generator semiring.

The first section recalls without proof some facts from polyhedral geometry needed in the paper. Except for the setting of a unified notation and for minor generalizations, the material there is well known. The second section defines PL maps and sets, and studies the category. The main results (leading to the classification theorem) are given in the last section.

1. Review of polyhedral geometry. The following conventions and definitions hold throughout. All vector spaces are finite-dimensional spaces over the reals $\boldsymbol{R}$; a *flat* means an affine submanifold of some such space V, and the *closed half-spaces* associated to a linear $f: V \to \boldsymbol{R}$ and an r in $\boldsymbol{R}$ (or associated to the hyperplane $\{x \mid f(x) = r\}$,) are the sets $\{x \mid f(x) \leqq r\}$ and $\{x \mid f(x) \geqq r\}$. The corresponding *open half-spaces* are obtained by using strict inequalities in the above. A *half-line* (closed or open) is the intersection of a line L in V with a (closed or open) half-space not containing L.

A (convex) closed *polyhedron* in V is by definition an intersection of finitely many closed half-spaces. The *dimension* of a nonempty polyhedron P is the dimension of aff (P), the smallest flat containing P; the *relative interior* ri (P) is the interior of P relative to the usual topology on aff (P). An *open polyhedron* P is by definition the relative interior of some closed polyhedron $c\ell(P)$ ($c\ell$ denoting