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UNIVERSAL CONNECTIONS: THE LOCAL PROBLEM

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It is well known that the canonical connection on the Stiefel bundle over the Grassman manifold is universal in a certain range of dimensions. We give some local results specifying necessary and sufficient conditions for the connection to be universal for particular dimensions.

1. Introduction. In this paper we consider connections on a principal G-bundle over a manifold M of dimension m, where G is $O(n)$, $U(n)$, or $\mathrm{Sp}(n)$. The universal examples are the bundles

$$\begin{gathered} O(N)/O(N-n) \longrightarrow O(N)/O(n) \times O(N-n) \\ U(N)/U(N-n) \longrightarrow U(N)/U(n) \times U(N-n) \\ \mathrm{Sp}(N)/\mathrm{Sp}(N-n) \longrightarrow \mathrm{Sp}(N)/\mathrm{Sp}(n) \times \mathrm{Sp}(N-n) \end{gathered}$$

with their canonical connections. (See § 2 for the definitions.) These are the Stiefel and Grassman manifolds for $\boldsymbol{R}$, $\boldsymbol{C}$, and $\boldsymbol{H}$. It is a theorem of Narasimhan and Ramanan that the canonical connection is universal in the sense that any connection on a principal G-bundle over M^m is induced by a map into the appropriate Grassman manifold N is sufficiently large. According to [2], [4], and § 8, it suffices to take

$$N \geqq 2n(m+1)(2mn^2+1) \qquad \text{or} \qquad \frac{1}{2}[(n+m)^2+7(n+m)+10]$$

for $O(n)$,

$$N \geqq n(m+1)(2mn^2+1)$$

for $U(n)$, and

$$N \geqq n(m+1)(4mn^2+2mn+1)$$

for $\mathrm{Sp}(n)$.

These inequalities are not sharp. The situation is analogous to the problem of finding isometric imbeddings of Riemannian manifold into Euclidean space. In that case, global results are available, but they are not sharp with respect to dimension. The only sharp results date back to L. Schläfli [3] who found the least dimensional Euclidean space for the local isometric imbeddability of a real-analytic Riemannian manifold. This was made more rigorous by M. Janet, C. Burstin, and E. Cartan. (See [5] for an account of this theorem.)

The principal result of this paper is to show how E. Cartan's theory of differential systems can be used to get local existence theorems for connection preserving maps into the appropriate Grassmanian. These results are sharp with respect to dimension.