PACIFIC JOURNAL OF MATHEMATICS
Vol. 98, No. 1, 1982

ELEMENTARY PROOFS OF BERNDT'S RECIPROCITY LAWS

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Using analytic functional equations, Berndt derived three reciprocity laws connecting five arithmetical sums analogous to Dedekind sums. This paper gives elementary proofs of all three reciprocity laws and obtains them all from a common source, a polynomial reciprocity formula of L. Carlitz.

1. Introduction. The classical Dedekind sums

$$s(h, k) = \sum_{r \bmod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right),$$

where h and k are integers, $k > 0$, $((x)) = x - [x] - 1/2$ if $x \neq$ integer, and $((x)) = 0$ for integer x, occur in the transformation formula for the logarithm of the Dedekind eta function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}) . \qquad (\mathrm{Im}\,(\tau) > 0)$$

Dedekind's formula which describes the behavior of $\log \eta(\tau)$ under a unimodular substitution implies a reciprocity law relating $s(h, k)$ and $s(k, h)$ when $(h, k) = 1$. (See [1], Chapter 3.)

Berndt [2] derived transformation formulas for the logarithm of the theta function

$$\theta(\tau) = \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})(1 + e^{(2n-1)\pi i\tau})^2$$

and related functions, and introduced five new arithmetical sums which are analogous to (but quite different from) the Dedekind sums, and showed that the analytic functional equations imply reciprocity laws for these sums. The sums in question are

$$(1) \qquad S(h, k) = \sum_{r=1}^{k-1} (-1)^{r+1+[hr/k]} ,$$

$$(2) \qquad s_1(h, k) = \sum_{r=1}^{k-1} (-1)^{[hr/k]}\left(\left(\frac{r}{k}\right)\right) ,$$

$$(3) \qquad s_2(h, k) = \sum_{r=1}^{k-1} (-1)^{r}\left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right) ,$$

$$(4) \qquad s_3(h, k) = \sum_{r=1}^{k-1} (-1)^{r}\left(\left(\frac{hr}{k}\right)\right) ,$$