PACIFIC JOURNAL OF MATHEMATICS
Vol. 98, No. 2, 1982

TOPOLOGICAL METHODS FOR C^*-ALGEBRAS II: GEOMETRIC RESOLUTIONS AND THE KÜNNETH FORMULA

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Let A and B be C^*-algebras with A in the smallest subcategory of the category of separable nuclear C^*-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\boldsymbol{Z}$ and by $\boldsymbol{R}$. Then there is a natural $\boldsymbol{Z}/2$-graded Künneth exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow \operatorname{Tor}(K_*(A), K_*(B)) \longrightarrow 0 .$$

Our proof uses the technique of geometric realization. The key fact is that given a unital C^*-algebra B, there is a commutative C^*-algebra F and an inclusion $F \to B \otimes \mathscr{K}$ such that the induced map $K_*(F) \to K_*(B)$ is surjective and $K_*(F)$ is free abelian.

1. Introduction. Let A and B be C^*-algebras. There is a $\boldsymbol{Z}/2$-graded pairing (defined in §2)

$$\alpha\colon K_p(A) \otimes K_q(B) \longrightarrow K_{p+q}(A \otimes B) \quad p,\, q \in \boldsymbol{Z}/2$$

where K_* denotes K-theory for Banach algebras [9, 17] and $\otimes = \otimes_{\min}$. Let $\mathfrak{N}$ be the smallest subcategory of the category of separable nuclear C^*-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\boldsymbol{Z}$ and by $\boldsymbol{R}$. We shall establish the following theorem.

KÜNNETH THEOREM. *Let A and B be C^*-algebras with $A \in \mathfrak{N}$. Then there is a natural short exact sequence*

$$0 \longrightarrow K_*(A) \otimes K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \xrightarrow{\beta} \operatorname{Tor}(K_*(A), K_*(B)) \longrightarrow 0 .$$

The sequence is $\boldsymbol{Z}/2$-graded with $\deg \alpha = 0$, $\deg \beta = 1$, *where $K_p \otimes K_q$ and* $\operatorname{Tor}(K_p, K_q)$ *are given degree $p + q$ ($p, q \in \boldsymbol{Z}/2$).*

If $A = C(X)$ and $B = C(Y)$ with X and Y finite CW-complexes then the hypotheses are satisfied and we recover the classical Künneth Theorem for topological K-theory due to Atiyah [1]: