PACIFIC IOURNAL OF MATHEMATICS
Vol. 98, No. 2, 1982

AN ERROR ESTIMATE UNIFORM IN TIME FOR SPECTRAL GALERKIN APPROXIMATIONS OF THE NAVIER-STOKES PROBLEM

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1. Introduction. The existence theory for the nonstationary Navier-Stokes equations can be developed, by the method of Galerkin approximation, using any of a wide variety of possible systems of basis functions. The basis functions used in the papers of Hopf [8] and of Kiselev and Ladyzhenskaya [10] are merely assumed to belong to and be complete in certain function spaces. However, to obtain refinements in the theory by the Galerkin method, particularly regarding the regularity and decay of solutions, it often appears essential to choose as basis functions the eigenfunctions of the Stokes operator; see Ito [9], Lions [12], Prodi [14], Foias [3], Ladyzhenskaya [11], Temam [17], and Heywood [4, 5, 6].

In these and other works, the convergence of the Galerkin approximations is generally proved by a compactness argument, based on *a*-priori bounds for the approximations. A notable exception is the paper [3] of Foias, where (on page 324) the approximations are shown to converge, uniformly over a time interval, in the Dirichlet norm. Recently, Rautmann [15, 16] has drawn attention to this type of result, and gone further, giving a systematic development of error estimates, for the Galerkin approximations and their time derivatives.

Rautmann's error estimates (and also Foias' convergence theorem) are presented locally, valid on a finite interval determined by certain norms of the data. At best, if one assumes the solution to be approximated is uniformly *regular*, for $t \in [0, \infty)$, the method yields an error estimate which grows exponentially with time. And, without further assumptions, this is the best that can be expected. However, if one assumes, additionally, the solution to be approximated is *stable*, then it is reasonable to expect an error estimate which is uniform in time. This is what is done in the present paper. It is hoped the result may prove suggestive for future developments in the Navier-Stokes theory. It should be mentioned, however, the original reason for undertaking this work arose in the author's joint study [7] with Rolf Rannacher, of finite element Galerkin approximations, a context in which error estimates uniform in time have important implications for computations. While the present work served to fix some ideas, in a simpler, purely theoretical context, it has turned out our arguments in the finite element context are substantially different.