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DUALITY CONDITION AND PROPERTY (S)

SU-SHING CHEN

We consider some geometric aspects of Borel's density theorem and property (S) of Selberg for simply connected complete Riemannian manifolds of nonpositive curvature. We also have some results on simply connected complete Kähler manifolds of nonpositive curvature.

A subgroup Γ of a topological group G is said to have property (S) in G if for each neighborhood U of the identity e of G and each element g in G there exists and integer $n > 0$ such that $g^n \in U \cdot \Gamma \cdot U$. In [3], Borel has proved the density theorem for subgroup Γ of property (S) in a connected semi-simple Lie group G without compact factors. Intuitively, it means that Γ is the product of some simple factors $\{G_i\}$ of $G = \prod_{i=1}^{k} G_i$ by a discrete group in the product of other simple $\{G_j\}$ of G (see p. 179 of [3]).

In [5], the duality condition for a group Γ of isometries of a simply connected complete Riemannian manifold M of nonpositive curvature was introduced. Γ satisfies the duality condition if for each infinite geodesic σ of M there is a sequence $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n(p) \to \sigma(\infty)$ and $\gamma_n^{-1}(p) \to \sigma(-\infty)$ for each p in M. If the quotient space M/Γ is compact or has finite volume, then Γ satisfies the duality condition [6], [7].

In this paper, we shall prove that if Γ is any subgroup of the isometry group $I(M)$ satisfying the duality condition and if M is a simply connected complete visibility manifold (see [6]) then either Γ is discrete or M is a rank one symmetric space of noncompact type and $(\bar{\Gamma})_0 = I_0(M)$ or $\bar{\Gamma}$ is of finite index (less than $[I(M), I_0(M)]$ in $I(M)$. This is an analogue of Borel's density theorem. In fact, the theorem is true if M satisfies a weaker condition of [1] and [8], that is, some geodesic σ of M does not bound an imbedded flat totally geodesic half plane. We shall compare the duality condition with the property (S). The duality condition is apparently weaker than property (S) for noncompact symmetric spaces.

In [12], Heintze has proved that a subgroup Γ of property (S) of the noncompact semi-simple Lie group G satisfies the duality condition. We shall prove that the duality condition is equivalent to a condition on the set of axial transformations (or transvections [18]) in G similar to the property (S). The last part of this paper is concerned with the complex version of several main theorems in [5]. These results seem to be interesting in the area of simply connected complete Kähler manifolds of nonpositive curvature investigated by