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ON THE PROXIMINALITY OF STONE-WEIERSTRASS SUBSPACES

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Let S be a compact Hausdorff space, X a Banach space, $C(S, X)$ the Banach space of all continuous X-valued functions on S equipped with the supremum norm. In this paper a necessary and sufficient condition on X for every Stone-Weierstrass subspace of $C(S, X)$ to be proximinal is established. Furthermore, it is shown that every such subspace is proximinal if X is a dual locally uniformly convex space.

Introduction and notations. Let S be a compact Hausdorff space, X a Banach space, $C(S, X)$ the Banach space of all continuous functions on S with values in X, equipped with the supremum norm. The purpose of this paper is to study the proximinality of certain subspaces, the so-called Stone-Weierstrass subspaces (SW-subspaces) of $C(S, X)$. This problem has been studied by many authors: Mazur (unpublished, cf., e.g., [11]) proved that every SW-subspace of $C(S, X)$ is proximinal if X is the real line $\boldsymbol{R}$ (a subspace G of a normed linear space Y is called proximinal if every $y \in Y$ possesses an element of best approximation x_0 in G, i.e., if there is an $x_0 \in G$ such that $\|y - x_0\| \leqq \|y - x\|$ holds for every $x \in G$). Pelczynski [9] and Olech [8] asked for which Banach spaces X every SW-subspace of $C(S, X)$ is proximinal. Olech [8] and Blatter [2] showed that this is true if X is a uniformly convex Banach space and an L_1-predual space, respectively. It has been shown in [6] that there exists a Banach space X and a compact Hausdorff space S such that $C(S, X)$ has a non-proximinal SW-subspace. Thus, the above mentioned question of characterizing those Banach spaces X for which every SW-subspace is proximinal, arises naturally. Here we give such a characterization. Using a modification of a method due to Olech [8], we show further that if X is a locally uniformly convex space such that every compact subset of X has a Chebychev center (a point x_0 is called a Chebychev center of a bounded set F if x_0 is the center of a "smallest" ball containing F) then every SW-subspace of $C(S, X)$ is proximinal. Every dual space, e.g., has the latter property [3].

We use the following notations. $\boldsymbol{R}$ and N will denote the set of all real numbers and the set of all positive integers, respectively. Let X be a Banach space, $x \in X$, $r > 0$. $B(x, r)$ will denote the closed ball in X with center x and radius r. A set-valued function Φ from a topological space S into 2^X is said to be upper Hausdorff semicon-