PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 1, 1982

THE CAUCHY PROBLEM AND ASYMPTOTIC DECAY FOR SOLUTIONS OF DIFFERENTIAL INEQUALITIES IN HILBERT SPACE

G. N. HILE AND M. H. PROTTER

1. Introduction. Let H be a real or complex Hilbert space and A an operator with domain D in H. We consider the differential operators

$$\frac{du}{dt} - Au \tag{1.1}$$

$$\frac{d^2u}{dt^2} - Au\ , \tag{1.2}$$

and we investigate the Cauchy problem for differential equations and inequalities in which (1.1) and (1.2) are the principal parts. In general, we shall suppose that A is a nonlinear, unbounded operator, neither symmetric nor antisymmetric, and dependent on t. In §§ 2 and 3 we consider the case where A is a linear operator.

Operators of the type in (1.1) were considered by Agmon and Nirenberg [1] who used a convexity argument to establish not only uniqueness theorems for the Cauchy problem but also maximal rates of decay as $t \to \infty$.

In § 2 we treat linear operators A which can be represented in the form $A = M + N$ where M is symmetric and N is antisymmetric. These hypotheses are used mainly for computational convenience. Instead of symmetry, the actual principal hypothesis on M is the inequality

$$\begin{aligned}\frac{d}{dt}\operatorname{Re}(M(t)u(t),\ u(t)) - 2\operatorname{Re}(M(t)u(t),\ u'(t))\\ \geqq -\gamma_3\|M(t)u(t)\|\,\|u(t)\| - \gamma_4\|u(t)\|^2\ ,\end{aligned} \tag{1.3}$$

where γ_3, γ_4 are positive constants. Thus the results of Section 2, when applied to differential operators A, are not restricted to those operators for which the principal part is self-adjoint. Furthermore, the condition of antisymmetry on N is easily relaxed. The arguments in § 2 are applicable almost without change if N satisfies either the inequality

$$\operatorname{Re}(N(t)u(t),\ u(t)) \leqq \gamma(t)\|u(t)\|^2$$

or

$$\operatorname{Re}(N(t)u(t),\ u(t)) \geqq -\gamma(t)\|u(t)\|^2$$