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PROJECTIVE COLORINGS

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All colorings of the points of (Desarguesian) projective planes in three colors so that no straight line contains points of all three colors are characterized in terms of the valuations of the field of coordinates. Generalizations to higher dimensions and applications to the Fundamental Theorem of Projective Geometry and the division of polygons into disjoint triangles of equal areas are given. We restrict our discussion to the Pappian (commutative coordinate field) case. For the general division ring case see [3].

1. Introduction.

M. Bognár (Budapest University) recently posed the following question: can the points of the real projective plane be colored in three colors (nontrivially) so that no line contains all three colors? Here nontrivial means that no color is confined to one line. For the affine plane this question had been answered in the affirmative in [6]. In this paper we give a complete characterization of these colorings and extend our results in various directions (other fields, higher dimensions, and curves of higher degree). The results involve a nontrivial blending of combinatorics and valuation theory. In § 5, we give a generalization of the fundamental theorem of projective geometry suggested by our results. In § 6 we extend results of [6] to the division of polygons into disjoint triangles of equal areas.

After this paper was written we learned of the work of D. S. Carter and A. Vogt [3]. We wish to acknowledge their priority, especially in the discovery of Theorems 1 and 5 of this paper. In view of our somewhat different approach we give the comparatively short proofs of these two theorems.

2. Colorings of projective planes.

Let $P_2(F)$ denote the projective plane over the field F. (For convenience we assume F to be commutative, although it does not appear necessary for our results. It is not clear, however, whether our results extend to the non-Desarguesian case.) We wish to color the points of $P_2(F)$ in three colors, say red, white, and blue, so that no line contains points of all three colors. To avoid trivialities we assume, of course, that each color is used at least once. One type of coloring can be obtained by coloring a point p red, say, and coloring each line through p (with p deleted) either solid white or solid blue, randomly. Another type, essentially the dual of the above, is obtained by