PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 1, 1982

ON THE REGULARITY UP TO THE BOUNDARY FOR SECOND ORDER NONLINEAR ELLIPTIC SYSTEMS

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It is proved the regularity up to the boundary of the uniformly Lipschitz-continuous weak solutions of a boundary value problem for the elliptic system

$$(1.1)\qquad -D_i a_i^r(x, u, Du) + \bar{a}^r(x, u, Du) = f^r\ ; \qquad r = 1, \cdots, m$$

from the Liouville properties of the system.

In (1.1) $u = \{u^r\}_{r=1,\cdots,m}$ is a vector function and $Du = \{D_i u^r\}_{\substack{i=1,\cdots,n\\ r=1,\cdots,m}}$ is its gradient. We write $D_i u^r = \partial u^r / \partial x_i$ and the summation convention is used throughout the paper.). We follow the ideas of our previous work (see [1-4]) where interior regularity was shown to be equivalent (in some sense) to the Liouville property (L) (see Definition 2.2). In the present paper, regularity up to the boundary is shown to be, essentially, equivalent to the previous (L) together with a certain "boundary" Liouville property (L^+) (see Definition 2.3).

2. Notation and assumptions. Let $\boldsymbol{R}^n$ be an n-dimensional Euclidean space; for $x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n) \in \boldsymbol{R}^n$ let $|x| = \max\{|x_i|;\ i = 1, \cdots, n\}$; further let $\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n;\ x_n > 0\}$; $\Omega = \{x \in \boldsymbol{R}^n_+;\ |x| < 1\}$; $\Gamma = \{x \in \boldsymbol{R}^n;\ |x'| < 1;\ x_n = 0\}$; $B(x_0, R) = \{x \in \Omega;\ |x - x_0| < R\}$; $\Gamma(x_0, R) = \overline{B(x_0, R)} \cap \Gamma$.

Let us denote

$$a(x, u, Du) = \{a_i^r(x, u, Du)\}_{\substack{i=1,\cdots,n\\ r=1,\cdots,m}}$$

$$\bar{a}(x, u, Du) = \{\bar{a}^r(x, u, Du)\}_{r=1,\cdots,m}$$

$$f(x) = \{f^r(x)\}_{r=1,\cdots,m}\ ,$$

where a, $\bar{a}$ are once continuously differentiable functions on $\bar{\Omega} \times \boldsymbol{R}^m \times \boldsymbol{R}^{nm}$, and $f \in [W^{1,p/2}(\Omega)]^m$ for some p, $p > n$.

REMARK. In what follows we omit the notation of the Cartesian product. So we write $f \in W^{1,p}(\Omega)$ instead of $f \in [W^{1,p}(\Omega)]^{mn}$ etc.

In this notation the system (1.1) can be rewritten as

$$(2.1)\qquad -\operatorname{div}(a(x, u, Du)) + \bar{a}(x, u, Du) = f(x)$$

on Ω. We suppose that the strong ellipticity condition holds:

$$(2.2)\qquad \frac{\partial a_i^r}{\partial \eta_j^s}(x, \xi, \eta)\zeta_i^r \zeta_j^s > 0$$