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THE THEORY OF AD-ASSOCIATIVE LIE ALGEBRAS

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A Lie algebra $\mathscr{L}$ is said to be ad-associative if the image of the adjoint representation of $\mathscr{L}$ on $\mathscr{L}$ is an associative algebra under composition. We show that every ad-associative Lie algebra is a quotient of a left commutative ($xyw = yxw$) associative algebra by a *Lie* ideal. We conclude that every ad-associative Lie algebra is solvable and every irreducible representation of a nilpotent, ad-associative Lie group is square integrable modulo its kernel. We also characterize the *HAT* algebras of Howe [2] in terms of associative algebra.

Let $\mathscr{L}$ be a finite dimensional Lie algebra over $\boldsymbol{R}$ and let ad denote the adjoint representation of $\mathscr{L}$ on $\mathscr{L}$. Let ad $\mathscr{L}$ be the image of ad. $\mathscr{L}$ is said to be ad-associative if ad $\mathscr{L}$ is closed under composition. In this case ad $\mathscr{L}$ is an associative algebra. Let us denote this algebra by $\mathscr{A}$. It is the purpose of this paper to give a structure theory for the ad-associative Lie algebras.

Our interest in the subject of ad-associative algebras stems from several different sources. The *HAT* algebras introduced by Howe in [2] in connection with the study of ocillatory integrals can be shown to be ad-associative. In fact, we prove what we feel to be a very pretty characterization of the *HAT* algebras. Ad-associative algebras also occur naturally in algebraic topology as a way of combining the information contained in the homology and co-homology groups of compact manifolds together. Here they give rise to some new topological invariants which are functions of the joint homology and cohomology groups (see Example II below). In another direction, there is a natural way of associating with any multi-linear form $\mathscr{B}$ on a vector space $\mathscr{V}$ an ad-associative Lie algebra $\mathscr{L}_{\mathscr{B}}$. Whether or not this association has any real significance remains to be seen. At the very least, the study of ad-associative algebras provides an interesting source of examples.

To begin our discussion, recall that any associative algebra $\mathscr{A}$ gives rise to a Lie algebra by setting $[x, y] = xy - yx$. If $\mathscr{L}$ is ad-associative, the Lie algebra corresponding to $\mathscr{A} = \text{ad}\,\mathscr{L}$, is just $\mathscr{L}/\mathscr{Z}$ where $\mathscr{Z}$ is the center of $\mathscr{L}$. Hence $\mathscr{A}$ fits into the exact sequence of Lie algebras $0 \to \mathscr{Z} \to \mathscr{L} \to \mathscr{A} \to 0$. As a vector space $\mathscr{L} = \mathscr{A} \times \mathscr{Z}$. In fact, there is an alternating, bi-linear mapping $\phi: \mathscr{A} \times \mathscr{A} \to \mathscr{Z}$ such that

$$[(x, s), (y, t)] = ([x, y], \phi(x, y))$$