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HOMOMORPHISMS OF MONO-UNARY ALGEBRAS

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Novotny has presented what amounts to a necessary and sufficient condition for the existence of certain homomorphisms between mono-unary algebras. In this paper, an example is presented to show that Novotny's condition is not sufficient, and a slightly stronger condition is shown to be both necessary and sufficient. The techniques of the proof are essentially the same as those used by Novotny.

Before proceeding, a brief summary is given of the relevant definitions.

A mono-unary algebra ("algebra" for short) is a pair (M, f) where f is any self-map of the set M; a homomorphism from (M, f) to (N, g) is a map $F: M \to N$ such that $F \circ f = g \circ F$.

Given such an algebra (M, f), let $f^0(x) = x$ for all $x \in M$, and $f^{n+1}(x) = f(f^n(x))$ for all $n \in \omega$; then $[x] = \{f^n(x) \mid n \in \omega\}$ is the subalgebra generated by x. For $x, y \in M$, define $x \rho y$ iff $[x] \cap [y] \neq \varnothing$ (equivalently, $x \rho y$ iff $f^m(x) = f^n(y)$ for some $m, n \in \omega$). This defines a congruence ρ on the algebra (M, f), the blocks of which are called the connected components of (M, f); if there is only one such component then the algebra is called connected. The connected components of M are each subalgebras of M, and a map from M into any algebra is a homomorphism iff its restriction to each connected component of M is a homomorphism. For this reason we need only consider homomorphisms from connected algebras M.

For each $x \in M$, either $f^m(x) = f^n(x)$ for some $m \neq n$, in which case $[x]$ is finite, or $[x]$ is infinite. If $[x]$ is finite, let $L(x)$ be the smallest natural number m with $f^m(x) = f^{m+k}(x)$ for some $k \neq 0$, and let $R(x)$ be the smallest natural number $k \neq 0$ with $f^{L(x)}(x) = f^{L(x)+k}(x)$. ($R(x)$ is the "rank" as defined by Novotny.) Then $f^m(x) = f^n(x)$ for $m < n$ implies $L(x) \leqq m$ and $R(x) \mid n - m$. If $[x]$ is infinite, define $L(x) = \infty$ and $R(x) = 0$. (Here, and in the remainder of the paper, ∞ is defined to be greater than every ordinal number, and we will use the convention that 0 is divisible by *every* natural number.)

Now (as in Novotny) define sets $M_\alpha \subseteqq M$ for ordinals α inductively as follows:

$$M_0 = \{x \in M \mid f^{-1}(x) = \varnothing\}$$
$$M_\alpha = \left\{x \in M - \bigcup_{\lambda<\alpha} M_\lambda \,\middle|\, f^{-1}(x) \subseteqq \bigcup_{\lambda<\alpha} M_\lambda\right\} .$$

Then the M_α are all pairwise disjoint, and for all $x \in M$, either