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COUNTER-EXAMPLES TO SOME CONJECTURES ABOUT DOUBLY STOCHASTIC MEASURES

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Some new types of doubly stochastic measures are constructed. Using measure preserving transformations, one can construct examples of nontrivial extreme doubly stochastic measures which are absolutely continuous with respect to another extreme doubly stochastic measure (disproving a conjecture by Feldman). By combinatorial arguments, one gets an extreme doubly stochastic measure that is not concentrated on a countable union of function graphs and whose support is the whole unit square.

0. Let I be the unit interval, m the ordinary Lebesgue measure on I. A probability measure μ on $I \times I$ is called *doubly stochastic*, if its marginal distributions coincide with m (i.e., $\mu(A \times I) = \mu(I \times A) = m(A)$ for each Borel set $A \subseteq I$). This is thought of as a continuous analogue of the notion of a doubly stochastic matrix (see [8] for a survey of results about doubly stochastic matrices). By a theorem of G. Birkhoff and von Neumann, the extreme points of the set of doubly stochastic matrices are the permutation matrices. The continuous analogue of a permutation matrix would be the graph of a bijective, measure preserving function, but it is easy to construct extreme points that are not of this type (a deep study, showing that for some purposes these special measures may well suffice has been given in [13]). On the other hand, all concrete examples of extreme doubly stochastic measures (e.d.s.m.) that can be found in the literature are concentrated on the graphs (or inverse graphs) of functions, mostly even linear functions and there was some common belief that any double stochastic measure must in some sense be made up from graphs (cf. the beginning of §2 of [2]). A functional analytic characterization of the e.d.s.m. has been given by Douglas [4] and Lindenstrauss [7]. Several authors have tried to generalize properties of permutation matrices to these measures, see e.g., [1], [2], [10]. One of the aims of this paper is to present some new constructions of e.d.s.m. which will also disprove some natural conjectures. The second construction yields a measure that is not concentrated on graphs. While the measures in the first construction are still concentrated on two graphs, it turns out that even in this case the geometric interrelations are much more complicated. In [2] the following conjecture (attributed to J. Feldman) was mentioned: if μ is an e.d.s.m. and if ν is a doubly stochastic measure which is