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BMO FROM DYADIC BMO

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We give new proofs of four decomposition theorems for functions of bounded mean oscillation by first obtaining each theorem in the easier dyadic case and then averaging the results of the dyadic decomposition over translations in $\boldsymbol{R}_m$.

1. Introduction. Let φ be a locally integrable real function on $\boldsymbol{R}^m$, let Q be a bounded cube in $\boldsymbol{R}^m$, with sides parallel to the axes, and let $|Q|$ be the Lebesgue measure of Q. Then

$$\varphi_Q = \frac{1}{|Q|}\int_Q \varphi dx$$

is the average of φ over Q. We say φ has *bounded mean oscillation*, $\varphi \in \mathrm{BMO}$, if

$$\|\varphi\| = \sup_Q \frac{1}{|Q|}\int_Q |\varphi - \varphi_Q| dx < \infty \ .$$

A *dyadic cube* is a cube of the special form

$$Q = \{k_j 2^{-n} < x_j < (k_j + 1)2^{-n}; 1 \leqq j \leqq m\}$$

where n and k_j, $1 \leqq j \leqq m$, are integers, and φ has *bounded dyadic mean oscillation*, $\varphi \in \mathrm{BMO}_d$, if

$$\|\varphi\|_d = \sup_{Q \text{ dyadic}} \frac{1}{|Q|}\int_Q |\varphi - \varphi_Q| dx < \infty \ .$$

Then clearly $\mathrm{BMO} \subset \mathrm{BMO}_d$ with $\|\varphi\|_d \leqq \|\varphi\|$, but BMO and BMO_d are not the same space; the function $\log|x_j|\chi_{\{x_j>0\}}$ is in BMO_d but not in BMO. In analysis BMO is more important than BMO_d because BMO is translation invariant, but BMO_d is not. On the other hand, BMO_d is very much the easier space to work with because dyadic cubes are nested (if two open daydic cubes intersect then one of them is contained in the other). For example, for BMO the original proofs [1], [6], [8], [11] of the four theorems stated below were rather technical, while for BMO_d the analogous results are comparatively trivial. In this paper we derive the four theorems from their dyadic counterparts.

Here is the idea. Let $T_\alpha\varphi(x) = \varphi(x - \alpha)$. Then

$$\varphi(x) = \lim_{N\to\infty} \frac{1}{(2N)^m}\int_{|\alpha_j|\leqq N} T_\alpha\varphi(x + \alpha) d\alpha \ .$$