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ISOPERIMETRIC EIGENVALUE PROBLEM OF EVEN ORDER DIFFERENTIAL EQUATIONS

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This paper is concerned with the following eigenvalue problem

$$(1)\qquad \begin{cases} x^{(2n)} + (-1)^{n+1}\lambda p(t)x = 0 \\ x^{(2k)}(0) = 0 = x^{(2k)}(1)\,, \qquad k = 0, 1, \cdots, n-1\,, \end{cases}$$

where $p(t)$ is assumed to be positive and continuous in $[0, 1]$. For the class of functions $q(t)$ which are equimeasurable to $p(t)$, we shall show that the rearrangement of $p(t)$ in symmetrically increasing order maximizes the least positive eigenvalue of (1), while the rearrangement of $p(t)$ in symmetrically decreasing order minimizes it.

Rearrangements of sets of numbers and functions are defined and investigated in detail in the book by Hardy, Littlewood and Pólya [11, Chapter X] and the book by Pólya and Szegö [18]. Using these notions, classes of nonhomogeneous strings, membranes, rods and plates with equimeasurable densities are considered in [3, 4, 5, 10] and the extremum of the principal frequencies are found for these classes. In particular, the above assertion has been proven by Beesack and Schwarz [5] and Fink [10] for $n = 1$. For $n = 2$, the proof is given by Banks [3]. Our proof will differ from those given for the special cases in that we will rely on some of the results in the theory of positive operators [12, 13, 14, 15, 16, 17] and certain rearrangement inequalities [18, 19]. All the required results will be explicitly stated in the sequel; the explanations of which, however, will be brief.

2. Rearrangement inequalities. Let h be a real function defined on a subset S of R^n, we shall denote the level set

$$\{t \in S: h(t) \geqq c\}$$

by $L(h, c)$. Two real functions $f(t)$ and $g(t)$ defined on $[0, 1]$ are called similarly ordered if, for each pair of points t_1, t_2 of $[0, 1]$, we have

$$[f(t_1) - f(t_2)][g(t_1) - g(t_2)] \geqq 0\ ;$$

f and g are called oppositely ordered if f and $-g$ are similarly ordered. If for each $c \in R$, the measure of $L(f, c)$ is equal to that of $L(g, c)$, then we say that f and g are equimeasurable. Let f, $\check{f}$ and $\hat{f}$ be equimeasurable, and in addition let $\check{f}(t)$ and $(2t-1)^2$ be